

Acoustic scattering by baffled membranes^{a)}

Gregory A. Kriegsmann, Andrew Norris, and Edward L. Reiss

Department of Engineering Sciences and Applied Mathematics, The Technological Institute, Northwestern University, Evanston, Illinois 60201

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A flexible membrane is set in an infinite plane baffle. The plane separates an acoustic field from a vacuum. A time harmonic wave is incident from the fluid on the membrane. When the frequency of the incident wave is not close to an *in-vacuo* resonant frequency of the membrane, the reaction of the fluid on the membrane is small. However, near a resonant frequency the fluid-membrane coupling is significant. We use the method of matched asymptotic expansions to obtain an asymptotic expansion of the scattered field. It is uniformly valid in the incident frequency. The expansion parameter $\epsilon \ll 1$ is the ratio of the fluid and membrane densities. The outer expansion, valid away from resonance, is $O(\epsilon)$. The inner expansion, valid near resonance, is of order unity. The fluid loading is shown to have the effect of decreasing the resonant frequencies from those of the *in-vacuo* membrane. Simple and double resonant frequencies are analyzed. However, the method is applicable to higher order resonant frequencies. Finally, the method is applied to normal incidence of a plane wave on a circular membrane.

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INTRODUCTION

A thin, tightly stretched membrane is set in an acoustically rigid infinite plane $Z = 0$. The upper half space is an acoustic fluid and the lower half space is a vacuum. A time harmonic wave propagating in the upper half space is incident on the plane. It scatters in the usual way from the rigid part of the plane as a specularly reflected wave. However, the field scattered by the membrane is more complex since the membrane deflects and oscillates in response to the incident field. In particular, when the frequency of the incident field is far from a resonance (natural) frequency of the membrane in a vacuum, the reaction of the fluid on the membrane is small. However, when the incident field frequency is close to a resonance frequency, the fluid and the membrane strongly couple and the amplitude of the membrane's motion and of the scattered field are large.

Such resonant interactions have been observed experimentally in other scattering problems in acoustic, elastic, and electromagnetic wave propagation.^{1,2} In addition, they have been studied for problems in which explicit representations can be obtained for the solutions, e.g., by "partial wave" expansions (see Ref. 1 for a review of previous investigations). Thus the previous analytical investigations have been restricted to simple "separable" geometries. In this paper we employ the method of matched asymptotic expansions to obtain asymptotic approximations of the scattered field, as the parameter $\epsilon \rightarrow 0$, that are uniformly valid in the frequency of the incident field. The small parameter ϵ is defined as the ratio of the fluid and membrane densities. The idea of solving scattering problems for flexible surfaces using this small parameter is due to Leppington.³ However, he employed a different method of analysis.

The outer expansion of the method of matched asymptotic expansions, which is $O(\epsilon)$, gives the scattered field away from the resonances. The inner expansions, which are $O(1)$, give the scattered fields at and near the resonances. The inner and outer expansions are then combined to obtain the composite, or uniformly valid asymptotic approximation to the scattered field. This approximation is expressed in terms of the *in-vacuo* normal modes of the membrane. The inner expansion is related to the classical resonant scattering approximation, for those problems which can be solved explicitly (see Ref. 1 and references given there for a discussion of resonant scattering theory and its applications).

A virtue of our method is that the leading term in the asymptotic expansion clearly reveals the precise structure of the solution and its dependence on the incident frequency without requiring an explicit representation, e.g., by partial waves. However, for its validity ϵ must be small, or equivalently the membrane must be heavy compared to the fluid. Furthermore, the results of our analysis suggest efficient numerical methods for evaluating the scattered fields, as we discuss in Sec. IV. The method is developed in Secs. II and III for resonant frequencies which are simple, i.e., there is only one *in-vacuo* normal mode of the membrane corresponding to that resonant frequency. Multiple resonant frequencies are analyzed in Sec. VI. The method is applied in Sec. V to the normal incidence of a plane wave on a circular membrane.

The analysis in this paper can be extended to other resonant scattering problems such as the scattering from a membrane that is backed by another acoustic fluid rather than a vacuum, the scattering from baffled elastic plates, and the scattering by nearly soft and nearly rigid three-dimensional acoustic targets. We expect to present the results of these investigations in future publications.

I. FORMULATION

We assume that the incident and scattered fields and the membrane's motion are proportional to $\exp(-i\omega t)$, where ω

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is the circular frequency of the incident field. This time factor is omitted in the subsequent analysis. Dimensionless space variables $\mathbf{x} \equiv (x, y, z)$ are defined by dividing the dimensional variables by a characteristic length L of the membrane, such as its maximum "diameter." Then the acoustic velocity potential $\Phi(\mathbf{x})$ satisfies the Helmholtz equation

$$\Delta_0 \Phi + k^2 \Phi = 0, \quad k \equiv \omega L / c_a \quad (1)$$

in the upper half-space $z > 0$. Here, c_a is the acoustic sound velocity, k is the dimensionless acoustic wavenumber, and Δ_0 is the Laplacian in $\mathbf{x}$. The acoustic pressure P is related to the potential by

$$P = \rho_a i \omega \Phi, \quad (2)$$

where ρ_a is the mass density of the acoustic fluid.

The equation of motion for the lateral deflection $w(x, y)$ of the membrane, which lies in the region M of the plane $z = 0$, is

$$\Delta w + k^2 c^2 w = \frac{L^2 P(x, y, 0)}{T}, \quad c \equiv \frac{c_a}{c_m} \quad (3)$$

Here, Δ is the Laplacian in x and y , $c_m \equiv (T/\rho_m)^{1/2}$, ρ_m is the density per unit area of the membrane, and T is the tension applied to the membrane. The acoustic pressure $P(x, y, 0)$ acts as a driving force on the membrane.

Since the plane $z = 0$ is acoustically rigid outside of M , we have the condition

$$\Phi_z(x, y, 0) = 0, \quad x, y \in M. \quad (4a)$$

Here the subscript denotes partial differentiation. In addition, the acoustic and membrane motions are coupled by the requirement that their vertical velocities are continuous on the membrane's surface. This gives

$$\Phi_z(x, y, 0) = -i(\omega L)w(x, y), \quad x, y \in M. \quad (4b)$$

We denote the potential of the incident acoustic field by $\phi = \Phi^I$. It is a solution of (1) and hence it depends on k . Then we express the total acoustic field in $z > 0$ by

$$\Phi(x, y, z) = \Phi^I(x, y, z; k) + \Phi^I(x, y, -z; k) + \phi(x, y, z). \quad (5)$$

In (5) $\Phi^I(x, y, -z; k)$ is the specularly reflected field from the rigid plane $z = 0$ and ϕ is the field scattered by the membrane. We do not indicate the dependence of ϕ on k . By inserting (5) into (2) and (4b) and then substituting the results into (1), (3), and (4a) we obtain, by eliminating w , the scattering problem for ϕ as

$$\Delta_0 \phi + k^2 \phi = 0, \quad \text{for } z > 0, \quad (6)$$

$$\phi_z(x, y, 0) = 0, \quad \text{for } x, y \notin M, \quad (7)$$

$$\Delta \phi_z(x, y, 0) + k^2 c^2 \phi_z(x, y, 0) = \epsilon k^2 c^2 [\phi(x, y, 0) + 2\Phi^I(x, y, 0; k)], \quad \text{for } x, y \in M. \quad (8)$$

In addition, ϕ satisfies the Sommerfeld radiation condition as $z \rightarrow \infty$. The dimensionless parameter ϵ in (8), which is defined by

$$\epsilon \equiv (\rho_a / \rho_m) L, \quad (9)$$

is the ratio of the volume densities of the fluid and the membrane. For many acoustic fluids and membrane materials ϵ is small. For example, for air and aluminum $\epsilon \approx 5 \times 10^{-4}$.

This formulation of the scattering problem is now sim-

plified by employing the Green's function $g(\mathbf{x}|\xi; k)$ of the Helmholtz equation (6) for the upper half space that satisfies the boundary condition $\phi_z(x, y, 0) = 0$ for all x and y and the radiation condition as $z \rightarrow \infty$. Then the scattered potential is given by

$$\phi(\mathbf{x}) = G(\mathbf{x}; k) v \equiv \iint_M g(\mathbf{x}|\xi, \eta, 0; k) v(\xi, \eta) d\xi d\eta, \quad (10)$$

where ξ is the vector with components (ξ, η, ξ) , g is given by

$$g(\mathbf{x}|\xi; k) = -(e^{ik|\mathbf{x} - \xi|}) / (2\pi|\mathbf{x} - \xi|), \quad (11)$$

and we have employed the notation

$$v(x, y) \equiv \phi_z(x, y, 0) \quad \text{for } x, y \in M. \quad (12)$$

Thus the scattering problem (6)–(8) is reduced to solving the integro-differential equation

$$\Delta v + k^2 c^2 v = \epsilon k^2 c^2 [G(k)v + 2\Phi^I(x, y, 0; k)], \quad (13a)$$

for $x, y \in M$ subject to the condition

$$v(x, y) = 0, \quad \text{on } B, \quad (13b)$$

where B is the boundary curve of the membrane. The boundary condition (13b) is obtained by continuity from (7) and (12). The integral operator $G(k)$ in (13a) is defined by

$$G(k) \equiv G(x, y, 0; k). \quad (13c)$$

It is proportional to the acoustic "back" pressure acting on the membrane's surface.

We observe that if the term Gv is omitted from (13a), the resulting boundary value problem is for the oscillations of a membrane in a vacuum driven by a force proportional to the incident field on the membrane. The dimensionless natural or resonant frequencies of the free oscillations of the membrane and the corresponding modes are proportional to the eigenvalues $k_n c$ and the corresponding eigenfunctions ψ_n of

$$\Delta v + k^2 c^2 v = 0, \quad v = 0 \quad \text{on } B. \quad (14)$$

Consequently, for the driven membrane with $k = k_n$, $n = 1, 2, \dots$ resonance occurs and the membrane's amplitude is unbounded if $\Phi^I(x, y, 0)$ is not orthogonal to ψ_n . However, for the scattering problem (13) with $k = k_n$ the back pressure Gv restrains resonance, and the amplitude is bounded. Moreover, if ϵ is small, as we assume, then the back pressure is small and a "near" resonance occurs.

In this paper we employ the method of matched asymptotic expansions⁴ to obtain an asymptotic approximation as $\epsilon \rightarrow 0$ of the solution of (13) that is uniformly valid in k . The corresponding acoustic field is then obtained from (10). For k bounded away from k_n (nonresonance) we solve (13) by a regular perturbation expansion in ϵ which is called the outer expansion. As $k \rightarrow k_n$ this outer expansion becomes singular, as we demonstrate. A second asymptotic expansion of the solution, which is called the inner expansion, is then obtained for k near k_n (near-resonance). The inner and outer expansions are then combined to form the composite expansion⁴ of the method of matched asymptotic expansions, which is the desired uniform asymptotic expansion. We should mention that in the traditional applications of the method of matched asymptotic expansions (see e.g., Refs. 5–7), the singularity, or boundary layer, occurs as the independent variables approach critical values. In the present appli-

cation the singularity in the expansion occurs when the parameter k approaches a critical value. This is similar to the application of the method to study imperfect bifurcation.⁸

II. THE UNIFORM ASYMPTOTIC EXPANSION

A. The outer expansion

For k bounded away from k_n , $n = 1, 2, \dots$, we seek an asymptotic expansion of the solution of (13) in the form

$$v = \sum_{j=0}^{\infty} v_j(x, y, k) \epsilon^j. \quad (15)$$

The coefficients v_j are determined by inserting (15) into (13) and equating coefficients of the same powers of ϵ in the result. Thus we obtain

$$\Delta v_j + k^2 c^2 v_j = k^2 c^2 [G(k) v_{j-1} + 2\Phi'(x, y, 0; k) \delta_{j1}] \\ v_j = 0 \text{ on } B, \quad j = 0, 1, 2, \dots, \quad (16)$$

where, $v_{-1} \equiv 0$ and δ_{j1} is the Kronecker δ function. Since $k \neq k_n$, we deduce from (16) with $j = 0$ that $v_0 \equiv 0$. Then inserting this result into (16) with $j = 1$, we solve the boundary value problem for v_1 by an expansion in the eigenfunctions ψ_n of (14). Assuming that the eigenfunctions form a complete orthonormal set, this gives,

$$v_1 = 2k^2 \sum_{j=1}^{\infty} \frac{\beta_j(k)}{k^2 - k_j^2} \psi_j(x, y), \quad (17a)$$

where the β_j are defined by

$$\beta_j(k) \equiv \langle \Phi'(x, y, 0; k), \psi_j(x, y) \rangle, \quad j = 1, 2, \dots, \quad (17b)$$

and we have used the notation

$$\langle f, g \rangle = \iint_M f(x, y) g(x, y) dx dy \quad (18)$$

for any two functions f and g defined on M . We can obtain the subsequent coefficients in the expansion (15) by solving the inhomogeneous problems (16) with $j = 2, 3, \dots$ with analogous eigenfunction expansions, but we do not present these results. In our analysis we assume that the eigenvalues and eigenfunctions of (14) are known either explicitly or from numerical computation.

The coefficient v_1 is unbounded as $k \rightarrow k_j$, $j = 1, 2, \dots$, as we observe from (17). Hence, the outer expansion

$$v = v_1 \epsilon + O(\epsilon^2) \quad (19)$$

is invalid for k at near the resonant frequencies of the membrane. Of course if $\beta_j(k_j) = 0$ then v_1 may be bounded as $k \rightarrow k_j$.

B. The inner expansion

We obtain an asymptotic expansion of the solution of (13) that is valid for k at and near these resonant frequencies by first defining the stretching parameter α by

$$k = k_n(1 + \alpha \epsilon) \quad (20)$$

for each fixed value of $n = 1, 2, \dots$. Then by inserting (20) into (13), we seek a second asymptotic expansion of the solution in the form

$$v = \sum_{j=0}^{\infty} V_j(x, y, \alpha) \epsilon^j. \quad (21)$$

Then we find, for example, that V_0 and V_1 are solutions of the boundary value problems:

$$\Delta V_0 + k_n^2 c^2 V_0 = 0, \quad V_0 = 0 \text{ on } B; \quad (22)$$

$$\Delta V_1 + k_n^2 c^2 V_1 = R_1 \equiv k_n^2 c^2 [-2\alpha V_0 + G(k_n) V_0 + 2\Phi'(x, y, 0; k_n)], \\ V_1 = 0 \text{ on } B. \quad (23)$$

The coefficients $V_2, V_3, \dots$ satisfy similar inhomogeneous boundary value problems.

Assuming that $k_n c$ is a *simple* eigenvalue of (14) with eigenfunction ψ_n , the solution of (22) is

$$V_0 = A_n \psi_n(x, y), \quad (24)$$

where the constant amplitude A_n is to be determined. Multiple eigenvalues will be considered in Sec. VI. We deduce from (23) that V_1 satisfies an inhomogeneous eigenvalue problem. Thus, R_1 in (23) must satisfy the solvability condition that it is orthogonal to every solution of the homogeneous problem corresponding to (22), i.e.,

$$\langle R_1, \psi_n \rangle = 0. \quad (25)$$

By inserting (24) for V_0 into the expression for R_1 , (25) is then reduced to a linear algebraic equation for the amplitude A_n . Its solution is

$$A_n = 2\beta_n(k_n)/(2\alpha + a_n), \quad (26)$$

where the a_n are defined by the fourfold integrals

$$a_n \equiv -\langle G(k_n) \psi_n, \psi_n \rangle \\ = -\iiint_M \int g(x, y, 0 | \xi, \eta, 0; k_n) \psi_n(\xi, \eta) \\ \times \psi_n(x, y) d\xi d\eta dx dy. \quad (27)$$

Thus, the inner expansion, which is valid for k near k_n , is given by

$$v = A_n \psi_n(x, y) + O(\epsilon). \quad (28)$$

C. The matching conditions

In the method of matched asymptotic expansions it is assumed that there exists an interval in k near k_n , which is called the overlap interval, and in which the outer and inner expansions are both valid asymptotic expansions of the solution. In this interval the outer and inner expansions must match⁴ in the following sense. Since the overlap interval is near k_n for small ϵ , we express the outer expansion in terms of the stretched (or inner) parameter α by inserting (20) into the outer expansion (19) and then re-expanding the result as

$$v = \sum_{j=0}^{\infty} \tilde{v}_j(x, y, \alpha) \epsilon^j. \quad (29)$$

Here, the $\tilde{v}_j$ are the outer coefficients expressed in terms of the inner parameter. For k in the overlap interval, we require that $|\alpha| \rightarrow \infty$ as $\epsilon \rightarrow 0$. Then, omitting all details, it can be shown that in this interval (29) is reduced to

$$v = [\beta_n(k_n)/\alpha] [1 + O(1/\alpha)] \psi_n + O(\epsilon). \quad (30)$$

Since in the overlap interval the outer expansion (29) and the inner expansion (21) are both valid asymptotic expansions of the solution, their difference must be asymptotic

to zero. We then deduce that the matching conditions⁴ are

$$\lim_{\alpha \rightarrow \infty} [V_j(x, y, \alpha) - \tilde{v}_j(x, y, \alpha)] = 0, \quad j = 1, 2, \dots \quad (31)$$

From the forms of the inner coefficients (28) and (26) and from (30) it can be shown that the conditions (31) are identically satisfied. We omit all details of this analysis.

D. The uniform expansion

The composite expansion of the method of matched asymptotic expansions provides the desired uniform asymptotic expansion of the solution for k in an interval about k_n . It is given by the sum of the inner and outer expansion minus the outer expansion in the inner parameter (30) in the overlap interval. Since in this interval the outer expansion $\rightarrow 0$ as $\alpha \rightarrow \infty$, to lowest order the composite expansion is

$$v \sim -\frac{\beta_n(k_n) a_n \psi_n}{\alpha(2\alpha + a_n)} + 2\epsilon k^2 \sum_{j=1}^{\infty} \frac{\beta_j(k) \psi_j}{k^2 - k_j^2}, \quad (32)$$

where α equals $(k - k_n)/\epsilon k_n$.

The uniform expansion for the scattered acoustic potential is obtained from (32), (12), and (10) as

$$\phi \sim \sum_{j=1}^{\infty} \left(-\frac{\beta_n(k_n) a_n \delta_{nj}}{\alpha(2\alpha + a_n)} + \frac{2\epsilon k^2 \beta_j(k)}{k^2 - k_j^2} \right) G(\mathbf{x}; k) \psi_j, \quad (33)$$

where $G(\mathbf{x}; k)$ is defined in (10). In the farfield, where $r = |\mathbf{x}| \rightarrow \infty$, we find by the standard application of the law of cosines and the binomial theorem to $|\mathbf{x} - \xi|$ in (11) that (33) becomes

$$\phi \sim \left[\sum_{j=1}^{\infty} \left(-\frac{\beta_n(k_n) a_n \delta_{nj}}{\alpha(2\alpha + a_n)} + \frac{2\epsilon k^2 \beta_j(k)}{k^2 - k_j^2} \right) F_j(k, \hat{\mathbf{r}}) \right] \frac{e^{ikr}}{r}. \quad (34)$$

The directivity factors F_l , $l = 1, 2, \dots$, are defined by

$$F_l(k, \hat{\mathbf{r}}) \equiv -\frac{1}{2\pi} \iint_M e^{-ik\hat{\mathbf{r}} \cdot \boldsymbol{\gamma}} \psi_l(\xi, \eta) d\xi d\eta. \quad (35)$$

Here, $\hat{\mathbf{r}} = \mathbf{x}/|\mathbf{x}|$ is the unit vector in the direction of observation, and $\boldsymbol{\gamma} = (\xi, \eta, 0)$. Thus in the farfield the composite expansion for the acoustic potential is reduced to a spherically outgoing wave whose amplitude is given by the sum in (34). The square of the absolute value of this sum is the differential cross section of the membrane corresponding to the composite expansion. The F_l are the Fourier transforms of the modes ψ_l with respect to the observation direction.

The asymptotic expansions presented in (32)–(34) are uniformly valid in an interval containing a simple eigenvalue k_n . They are valid for any other k bounded away from the other eigenvalues $k = k_j$ ($j \neq n$). If each eigenvalue is simple, then we obtain an asymptotic expansion that is valid for all k by replacing α by $(k - k_n)/\epsilon k_n$ in (32)–(34), and then summing over n . We find for example that the displacement v becomes

$$v \sim \sum_{m=1}^{\infty} \left(\frac{-k_m^2 a_m \beta_m(k_m) \epsilon^2}{(k - k_m)[2(k - k_m) + \epsilon k_m a_m]} + \frac{2\epsilon k^2 \beta_m(k)}{k^2 - k_m^2} \right) \psi_m(x, y). \quad (36)$$

Similarly the composite expansion of the farfield, corresponding to (34), that is valid for all k given by

$$\phi \sim A(\hat{\mathbf{r}}, k) (e^{ikr}/r), \quad (37a)$$

$$A(\hat{\mathbf{r}}, k) = \sum_{m=1}^{\infty} \left(\frac{-\epsilon^2 \beta_m(k_m) a_m k_m^2}{(k - k_m)[2(k - k_m) + \epsilon k_m a_m]} + \frac{2\epsilon k^2 \beta_m(k)}{k^2 - k_m^2} \right) F_m(k, \hat{\mathbf{r}}), \quad (37b)$$

where $|A(\hat{\mathbf{r}}, k)|^2$ is the scattered differential cross section of the membrane. A similar result holds for (33).

III. INTERPRETATION OF THE RESULTS

The inner and outer expansions can be recovered from the uniform expansions (32)–(34) by taking appropriate limits in these equations. Thus if $k - k_n = O(1)$ as $\epsilon \rightarrow 0$, so that $\alpha = O(1/\epsilon)$, then the first term of (32) is $O(\epsilon^2)$. Consequently, the second term dominates and (32) is reduced to the outer expansion (19). Similarly, the farfield potential given by (34) is reduced in the outer limit to

$$\phi \sim \left(2\epsilon k^2 \sum_{j=1}^{\infty} \frac{\beta_j(k) F_j(k, \hat{\mathbf{r}})}{k^2 - k_j^2} \right) \frac{e^{ikr}}{r}. \quad (38)$$

Thus the displacement v and the scattered potential ϕ are $O(\epsilon)$ when the incident frequency is bounded away from all of the membrane's resonant frequencies. That is, the acoustic potential is given essentially by the sum of the incident and specularly reflected waves because the acoustic fluid density is much smaller than the membrane density. This qualitative behavior has already been observed¹ in other scattering problems that can be solved explicitly, e.g., by partial wave expansions and from their subsequent numerical evaluations.

However, when the incident frequency approaches a resonant frequency, i.e., when $k = k_n(1 + \alpha\epsilon)$ for $\alpha = O(1)$ as $\epsilon \rightarrow 0$, the second term in (32) is $O(1)$. It combines with the first term to yield the inner result (28). Similarly, the farfield expression (34) for the scattered potential is reduced for k near k_n to

$$\phi \sim A_n F_n(k_n, \hat{\mathbf{r}}) (e^{ik_n r}/r). \quad (39)$$

This is $O(1/\epsilon)$ larger than the outer expansion (38) and it is of the same order as the incident and specularly reflected waves. Thus the scattering potential contributes to the lowest order approximation only when k is near a resonant frequency.

The coefficient of the outgoing spherical wave in (39) is the product of the amplitude A_n and the directivity factor F_n which gives the radiation pattern of the membrane for k near k_n . Furthermore, $|A_n F_n|^2$ is the differential cross section of the scattered acoustic potential for k near k_n and $|F_n(k_n, \hat{\mathbf{r}})|^2$ is the differential cross section of the farfield scattered acoustic potential ϕ_n produced by the membrane vibrating with frequency k_n and mode $\psi_n(x, y)$. The amplitude A_n contains information about the coupling between the acoustic medium and the membrane, which we now describe.

In Appendix A we show that the fourfold integral (27) which defines a_n is given by

$$a_n = R + iI, \quad (40)$$

where R and I are defined by

$$R \equiv \int \int \int_{z>0} (|\nabla \phi_n|^2 - k_n^2 |\phi_n|^2) dx dy dz, \quad (41)$$

$$I \equiv k_n \int_0^\pi \int_0^{\pi/2} |F_n(k_n, \hat{r})|^2 \sin \phi' d\phi' d\theta' > 0.$$

Here I/k_n is the total cross section of ϕ_n and R is twice the corresponding dimensionless Lagrangian. The total acoustic scattering cross section for k near k_n can therefore be written as

$$\sigma_T = (|A_n|^2/k_n)I. \quad (42)$$

Combining (26) and (40) we obtain

$$|A_n|^2 = \frac{4|\beta_n(k_n)|^2}{(2\alpha + R)^2 + I^2}. \quad (43)$$

The square of the modulus of A_n is sketched as a function of α in Fig. 1. Since the maximum occurs at $\alpha = -R/2$ and since $R > 0$, as we show in Appendix B, (20) implies that the maximum occurs for k slightly less than k_n . Thus R is a detuning parameter. Furthermore, it follows from (43) that I is the bandwidth of $|A_n|^2$. To evaluate $|A_n|^2$ it is assumed that k_n and $\psi_n(x, y)$ are known explicitly or by numerical computation. Then the integrals that define β_n , R , and I must be determined similarly.

If $\Phi' \equiv 0$, then $\beta_n(k_n) = 0$ and the solvability condition (25) gives

$$(2\alpha + a_n)A_n = 0. \quad (44)$$

A nonzero solution of this equation requires $\alpha = -\frac{1}{2}a_n$. From (20), (40), and (41) it follows that the complex eigenfrequency of the coupled fluid-elastic system is given by

$$k \sim k_n \left[\left(1 - \frac{\epsilon}{2} R \right) - \frac{i\epsilon}{2} I \right], \quad (45)$$

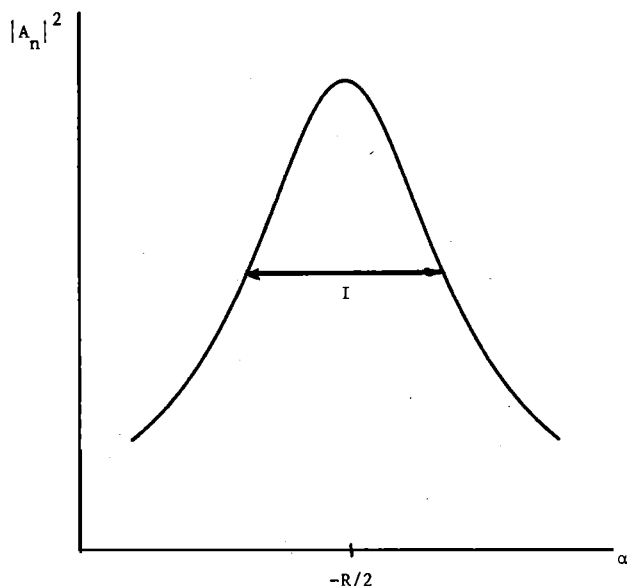


FIG. 1. A graph of $|A_n|^2$ for a simple eigenvalue.

to order $O(\epsilon)$. The negative imaginary part in (45) corresponds to damping because $e^{-i\omega t}$ is the assumed time dependence. Since the decay rate is proportional to I , this parameter measures the ability of the membrane vibrating at frequency k_n , to convert motion into acoustic energy.

IV. NUMERICAL PROCEDURES

When solving the scattering problem numerically by the method of normal modes, the membrane displacement is expressed as

$$v(x, y) = \sum_{m=1}^{\infty} b_m \psi_m(x, y). \quad (46)$$

Then by substituting (46) into (13) we find that the coefficients b_m are the solution of the infinite system of linear algebraic equations

$$\tau(k, \epsilon) \mathbf{b} = 2\epsilon k^2 \boldsymbol{\beta}(k), \quad (47)$$

where $\mathbf{b}$ and $\boldsymbol{\beta}$ are the infinite vectors with components $(b_1, b_2, \dots)$ and $(\beta_1, \beta_2, \dots)$, respectively, and τ is the symmetric infinite matrix whose components τ_{ij} are defined by

$$\tau_{ij}(k, \epsilon) \equiv (k^2 - k_j^2) \delta_{ij} + \epsilon k^2 D_{ij}(k), \quad i, j = 1, 2, \dots \quad (48)$$

The elements of the complex valued symmetric D_{ij} are defined by the fourfold integrals

$$D_{ij}(k) \equiv - \int \int \int \int_{MM} g(x, y, 0 | \xi, \eta, 0; k) \psi_i(\xi, \eta) \times \psi_j(x, y) d\xi d\eta dx dy, \quad (49)$$

so that $D_{nn}(k_n) = a_n$.

To solve (47) numerically we first truncate it to an $N \times N$ system. The value of N that is selected depends upon the value of k , the desired accuracy, and the size of the computer's memory. In general, the first N eigenvalues and eigenfunctions of the membrane are obtained by solving (14) numerically. Then the matrix elements D_{ij} are obtained by numerical integration in (49) using these eigenfunctions. Since the matrix D_{ij} is symmetric, only $N(N+1)/2$ of the elements need be computed. The truncated vector $\boldsymbol{\beta}(k)$ is also computed by numerical integration from the definition (17b). Finally, the truncated version of (47) must be inverted to obtain the approximate solution vector $\mathbf{b} = (b_1, b_2, \dots, b_N)$. These coefficients are inserted into (46) to yield the approximate solution. This process must be repeated for each desired value of k . We now show how the asymptotic method of Sec. II as applied to the algebraic system (47)–(49) can be used to substantially simplify and reduce the computations as compared to the direct numerical method described in (46)–(49) when ϵ is small. If k is bounded away from k_j ($j = 1, 2, \dots$), then the outer expansion given by (17)–(19) is valid. The infinite sum in (17) must be truncated at $n = N$. To evaluate this sum, the first N eigenvalues, eigenvectors, and the truncated vector $[\beta_1(k), \dots, \beta_N(k)]$ must be computed numerically for each desired value of k as in the direct numerical method. However, it is now not necessary to perform the costly numerical evaluations of the matrix elements $D_{ij}(k)$ and the subsequent numerical inversions for each desired

value of k because the k dependence of the solution is essentially given by the outer expansion.

If k is close to the simple eigenvalue k_n , then v is given asymptotically by V_0 in (24). The determination of V_0 only requires the numerical calculation of $\psi_n, \beta_n(k_n)$ which have already been obtained for the outer expansion, and $D_{nn}(k_n) = a_n$. The numerical savings here is even more substantial than in the outer region because the k dependence and the resulting matrix inversion are given explicitly by the inner expansion.

The numerical approximations to v in the inner and outer regions can now be combined to produce the uniform result given by (32). This result will become invalid near k_l for $l \neq n$. If all the eigenvalues of the membrane are simple, then (36) holds. Thus, only the matrix elements D_{ll} for $l = 1, 2, \dots, N$ need be evaluated numerically at $k = k_l$. Even in this more complicated expansion a sizable numerical advantage is gained. This is because the off-diagonal D_{mn} are not needed and no inversions are necessary.

V. NORMAL INCIDENCE OF A PLANE WAVE ON A BAFFLED CIRCULAR MEMBRANE

We now apply the asymptotic method to study the scattering of a normally incident plane wave,

$$\Phi^I(x, k) = e^{-ikx} \quad (50)$$

on a circular membrane. Then the membrane displacement and the scattered acoustic potential vary only with z and the cylindrical radius $\rho = \sqrt{x^2 + y^2}$. The axisymmetric eigenvalues and eigenfunctions of the unit circular membrane are⁹

$$k_n = \frac{\lambda_n}{c}, \quad \psi_n = \sqrt{\pi} \frac{J_0(ck_n \rho)}{J_0'(k_n)}, \quad n = 1, 2, \dots, \quad (51)$$

where λ_n is the n th root of $J_0(\lambda) = 0$. Since they are all simple eigenvalues the uniform expansions (36) and (37) are valid. To evaluate them, we must determine R and I which are defined in (41).

It follows from the Fourier analysis in Appendix B that R and I can be expressed as

$$R = \iint_{\gamma > k_n} \frac{|F_n(1, \gamma)|^2}{(\gamma^2 - k_n^2)^{1/2}} d\xi d\eta, \quad (52)$$

$$I = \iint_{\gamma < k_n} \frac{|F_n(1, \gamma)|^2}{(k_n^2 - \gamma^2)^{1/2}} d\xi d\eta. \quad (53)$$

Here γ is the two-dimensional vector with components (ξ, η) and $\gamma = |\gamma|$. We find from the definition of F_n in (35) and the properties of $J_0(ck_n \rho)$ that

$$F_n(k, t) = \frac{ck_n}{\sqrt{\pi}} \frac{J_0(kt \sin \Omega)}{(k^2 t^2 \sin^2 \Omega - c^2 k_n^2)}, \quad (54)$$

where t is an arbitrary vector, $t \equiv |t|$, and Ω is the angle between t and the z axis. Inserting (54) with $k = 1$, $t = \gamma$, and $\Omega = \pi/2$ into (52) and (53) we obtain the one-dimensional integrals

$$R = \frac{2c^2}{k_n} \int_0^\infty \left(\frac{J_0(k_n \cosh t)}{\cosh^2 t - c^2} \right)^2 \cosh t dt, \quad (55)$$

$$I = \frac{2c^2}{k_n} \int_0^{\pi/2} \left(\frac{J_0(k_n \cos \theta)}{\cos^2 \theta - c^2} \right)^2 \cos \theta d\theta. \quad (56)$$

Thus we have reduced the fourfold integrals in the definitions of R and I in (41) to the one-dimensional integrals (55) and (56). These integrals are numerically evaluated for a fixed n and varying c , where $k_n = \lambda_n/c$. They are graphed in Figs. 2 and 3 for $n = 1, \dots, 15$ and for $c < 1$.

The monostatic cross section of the scattered acoustic field is given by the value of $|A|^2$ in (37) in the backscattered direction. It is obtained by setting $\mathbf{t} = \hat{\mathbf{r}} = (0, 0, 1)$ in (54) and inserting this result into (37). We have summed this series using the numerical values of R and I given in Figs. 2 and 3 and the fact that for the circular membrane

$$\beta_n(k) = -2\sqrt{\pi}/ck_n, \quad \text{for all } k. \quad (57)$$

The square root of the monostatic cross section is graphed in Fig. 4 for $c = 0.5$ and for k in the range $0 < k < 20$. For this range of incident frequencies there are three resonant frequencies k_1, k_2 , and k_3 as is clearly indicated in the figure. The maxima occur at approximately $k = k_1, k_2$, and k_3 because the R are relatively small. In addition, the cross section is small $[O(\epsilon)]$ between the resonant peaks and only one minimum occurs between each pair of maxima. These minima, which are called antiresonances,¹⁰ occur close to the resonant frequencies as seen in Fig. 4.

The antiresonances do not occur in monostatic cross sections computed from either the inner or outer expansions separately. They apparently occur at frequencies where neither of these expansions is valid separately so that the uniform expansion must be employed. In fact, an analysis of the uniform expansion, which we do not present, shows that the antiresonances occur when $\text{Re } A = 0$. In addition, the analysis shows that at the antiresonance points $|A| = O(\epsilon^2)$.

We have also computed numerically the total scattering cross section of the circular membrane for the same range of incident frequencies using the uniform expansion (37). The logarithm of this quantity is graphed in Fig. 5. We observe that the local minima are located nearly symmetrically between the maxima, and antiresonances do not occur. This is in contrast to previous results¹⁰ where antiresonances in

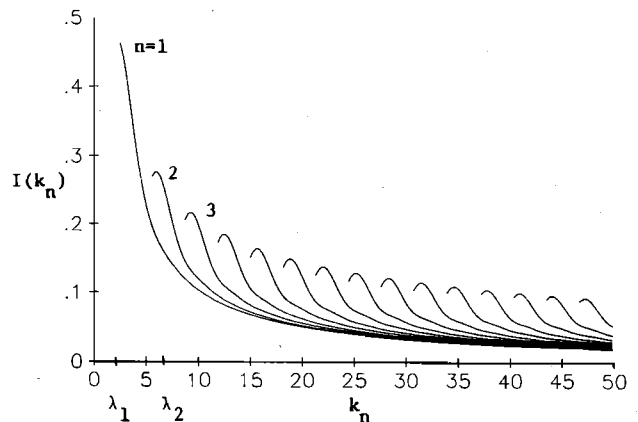


FIG. 2. A graph of the first 15 I as a function of $k_n = \lambda_n/c$, for $c < 1$.

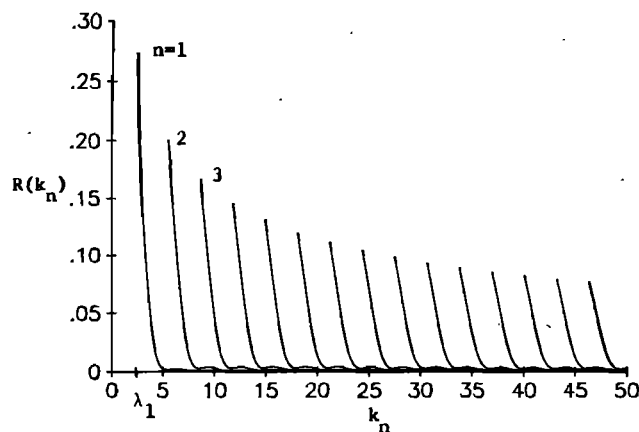


FIG. 3. A graph of the first 15 R as a function of $k_n = \lambda_n/c$, for $c < 1$.

the total scattering cross section were obtained. This discrepancy in the qualitative features of the solution may result from the different values of ϵ employed in evaluating the cross section in this paper ($\epsilon = 0.1$) and the values used in Ref. 10, where $\epsilon > 0.5$. However, it may also result from inaccuracies in the variational method¹⁰ where only two trial functions (modes) were employed. If the antiresonances occur in a region where the uniform expansion is required, as our results suggest, then many modes may contribute to values of the cross section as is suggested by the uniform expansion.

VI. MULTIPLE EIGENVALUES

The inner expansion obtained in Sec. II is for kc near the simple eigenvalue $k_n c$. However, membranes typically have eigenvalues with multiplicities greater than one, as well as simple eigenvalues. For example, the eigenvalues $k_{m,n} c$ and eigenfunctions $\psi_{m,n}$ of the unit square membrane are, $k_{m,n} c = \pi(m^2 + n^2)^{1/2}$, $\psi_{m,n} = 2 \sin m\pi x \sin n\pi y$, $m, n = 1, 2, \dots$. The lowest eigenvalue is $k_{1,1} c = \sqrt{2}$ and it is simple. However, the second eigenvalue $k_{1,2} c = k_{2,1} c = \sqrt{5}\pi$ has multiplicity two because $\psi_{1,2}$ and $\psi_{2,1}$ are linearly independent and they are the only eigenfunctions corresponding to this eigenvalue. Larger eigenvalues may have multiplicities > 2 .

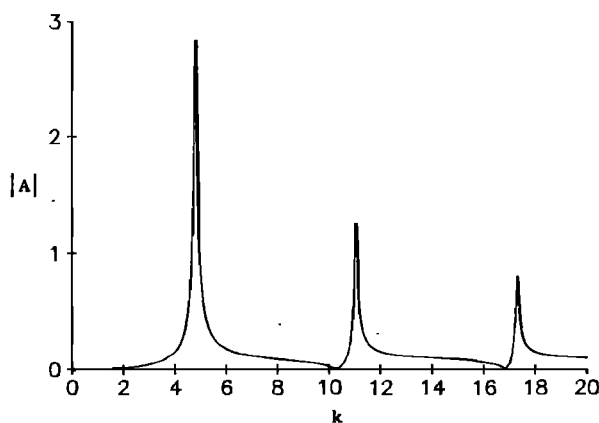


FIG. 4. Magnitude of the backscattered amplitude A for normal incidence on a circular membrane with $c = 0.5$ and $\epsilon = 0.1$.

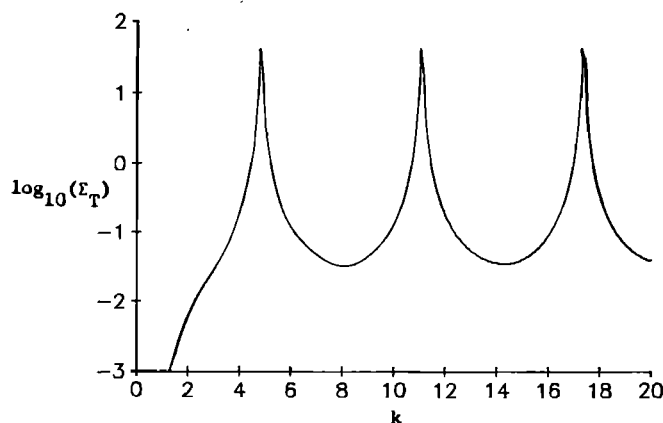


FIG. 5. Total scattering cross section Σ_T for $c = 0.5$ and $\epsilon = 0.1$.

We now obtain the inner expansion (21) for $k_n c$, an eigenvalue of multiplicity two. A similar analysis which we do not present applies to eigenvalues of higher multiplicity. Denoting the corresponding orthonormal eigenfunctions by ψ_n^1 and ψ_n^2 , the solution of (22) gives the inner expansion,

$$v = V_0 + O(\epsilon) = [A_n \psi_n^1 + B_n \psi_n^2] + O(\epsilon), \quad (58)$$

where the constant amplitudes A_n and B_n are to be determined. The solvability condition for (23) yields

$$\langle R_1, \psi_n^i \rangle = 0, \quad i = 1, 2. \quad (59)$$

By inserting V_0 from (58) into the expression for R_1 , (59) is reduced to the following linear algebraic equations for A_n and B_n

$$(2\alpha + T_{11})A_n + T_{12}B_n = 2\beta_n^1(k_n), \quad (60)$$

$$T_{21}A_n + (2\alpha + T_{22})B_n = 2\beta_n^2(k_n),$$

where

$$\beta_n^i(k_n) \equiv \langle \psi_n^i, \Phi^1(x, y, 0; k_n) \rangle, \quad i = 1, 2, \quad (61)$$

and the T_{ij} are defined for $i, j = 1, 2$ by the fourfold integrals

$$T_{ij} \equiv \langle \psi_n^i, G(k_n) \psi_n^j \rangle. \quad (62)$$

It follows from (62) that $T_{12} = T_{21}$.

The determinant $\Delta(\alpha)$ of the system (60) is the quadratic function given by

$$\Delta(\alpha) = (2\alpha + T_{11})(2\alpha + T_{22}) - (T_{12})^2 \equiv \Delta_R + i\Delta_I, \quad (63a)$$

where Δ_R and Δ_I are defined by

$$\Delta_R(\alpha) \equiv 4\alpha^2 + 2\alpha \operatorname{Re}(T_{11} + T_{12}) + \operatorname{Re}(T_{11}T_{22} - T_{12}^2), \quad (63b)$$

$$\Delta_I(\alpha) = 2\alpha \operatorname{Im}(T_{11} + T_{12}) + \operatorname{Im}(T_{11}T_{22} - T_{12}^2).$$

Since we prove in Appendix C that $\Delta(\alpha) \neq 0$ for all values of α , the solution of (60) is

$$A_n = [2\beta_n^1(k_n)(2\alpha + T_{22}) - 2T_{12}\beta_n^2(k_n)]/\Delta(\alpha), \quad (64)$$

$$B_n = [2\beta_n^2(k_n)(2\alpha + T_{11}) - 2T_{12}\beta_n^1(k_n)]/\Delta(\alpha).$$

To analyze the qualitative features of the inner expansion (58), we consider the magnitude $\langle v, v \rangle$. Then we obtain from (58),

$$\|v\|^2 = (A_n^2 + B_n^2) + O(\epsilon) = P(\alpha)b^2 + O(\epsilon), \quad (65)$$

where

$$P(\alpha) \equiv 4[(4\alpha^2 + 4\lambda\alpha + \gamma^2)/(\Delta_R^2 + \Delta_I^2)],$$

$$b^2 \equiv |\beta_n^1(k_n)|^2 + |\beta_n^2(k_n)|^2, \quad (66)$$

$$\gamma^2 \equiv [|T_{22}\beta_n^1(k_n) - T_{12}\beta_n^2(k_n)|^2 + |T_{11}\beta_n^2(k_n) - T_{12}\beta_n^1(k_n)|^2] (b^2)^{-1}$$

$$\lambda \equiv \{ \text{Re}(T_{22})|\beta_n^1(k_n)|^2 + \text{Re}(T_{11})|\beta_n^2(k_n)|^2 - 2 \text{Re}(T_{12})\text{Re}[\beta_n^1(k_n) \bar{\beta}_n^2(k_n)] \} (b^2)^{-1}.$$

The overbar denotes the complex conjugate. The quantity b is proportional to the projection of the energy of the incident wave into the null space of (22). Thus $\sqrt{P(\alpha)}$ is an "amplitude ratio" of the inner response (58), to lowest order in ϵ .

From (63) and (66) we deduce that

$$P(\alpha) = N_2(\alpha)/D_4(\alpha), \quad (67)$$

where $N_2(\alpha)$ and $D_4(\alpha)$ are quadratic and quartic polynomials, respectively. Both polynomials have real coefficients and are positive because $\langle v, v \rangle > 0$. The stationary points of $P(\alpha)$ are the roots of the quintic equation

$$D_4^2 P' = D_4(\alpha)N_2'(\alpha) - N_2(\alpha)D_4'(\alpha) = 0. \quad (68)$$

The quintic is not readily solved because it is a complicated function of T_{ij} and $\beta_n^j(k_n)$. However, (68) has real coefficients, so that it has either one, three, or five real roots. If it has one real root, the sketch of $P(\alpha)$ is similar to that of Fig. 1 which is the inner response for a simple eigenvalue. We have sketched $P(\alpha)$ in Figs. 6 and 7 when (68) has three and five roots, respectively. The maxima of $P(\alpha)$ and their corresponding local bandwidths may differ substantially, depending on the membrane's shape and on the incident field. In some cases, the local bandwidths may be narrow so that it

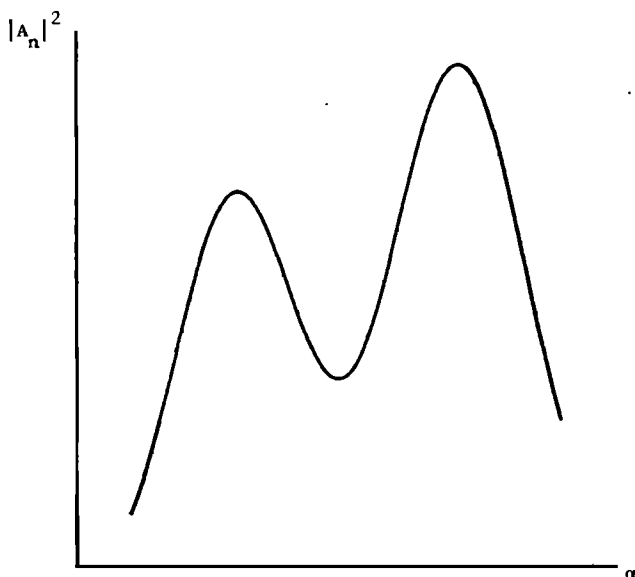


FIG. 6. A graph of $P(\alpha)$ for a double eigenvalue showing two relative maxima.

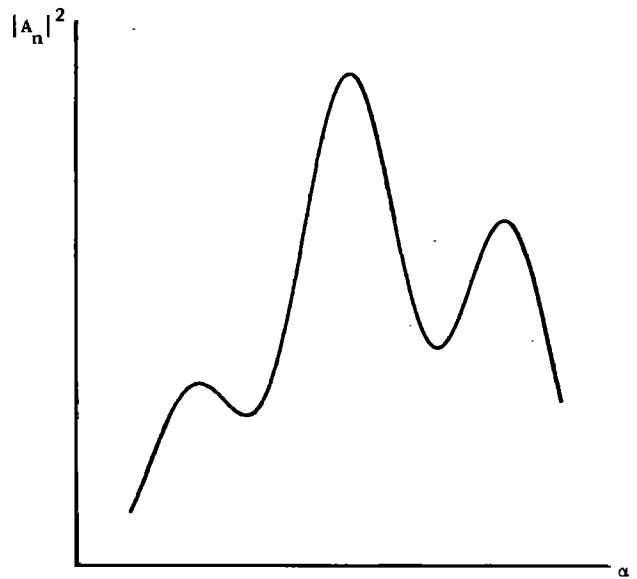


FIG. 7. A graph of $P(\alpha)$ for a double eigenvalue showing three relative maxima.

may be difficult to detect the double and triple maxima of the response in an experiment.

The inner expansion for the scattered acoustic potential corresponding to (58) which is obtained from (10)–(12) and (58) is

$$\phi = G(x, k_n)(A_n \psi_n^1 + B_n \psi_n^2) + O(\epsilon). \quad (69)$$

In the farfield as $r \rightarrow \infty$ (69) is reduced to

$$\phi = S(k_n, \hat{r})(e^{ik_n r}/r) + O(\epsilon), \quad (70a)$$

where $|S(k_n, \hat{r})|^2$ is the differential cross section of the membrane corresponding to the inner expansion. It is defined by

$$S(k_n, \hat{r}) \equiv A_n F_n^1(k_n, \hat{r}) + B_n F_n^2(k_n, \hat{r}), \quad (70b)$$

where F_n^j is defined by (35) with ψ_n replaced by ψ_n^j . The farfield inner potential given in (70) is a generalization to the double eigenvalue of the farfield potential given in (39) for the simple eigenvalue. It is a linear combination of the farfield inner potentials corresponding to each of the eigenfunctions of the double eigenvalue. The differential cross section (70b), is, in general, a complicated function of $\hat{r}$ because it depends upon the Fourier transforms F_n^j .

By employing the results in this section, we can show after a lengthy analysis (all details are omitted) that the total cross section Σ_T corresponding to (70) is given by

$$\Sigma_T(\alpha) = 4P_\Sigma(\alpha)b^2. \quad (71)$$

The amplitude ratio P_Σ is defined by

$$P_\Sigma(\alpha) \equiv k_n^{-1} \left(\frac{(2\alpha + \lambda)\Delta_I(\alpha) + \Gamma\Delta_R(\alpha)}{\Delta_R^2(\alpha) + \Delta_I^2(\alpha)} \right), \quad (72a)$$

where Γ is defined by

$$\Gamma \equiv \text{Im}(T_{22})|\beta_n^1(k_n)|^2 + \text{Im}(T_{11})|\beta_n^2(k_n)|^2 - 2 \text{Im}(T_{12})\text{Re}[\beta_n^1(k_n) \bar{\beta}_n^2(k_n)]. \quad (72b)$$

We observe that $P_\Sigma(\alpha)$ has the same mathematical structure as $P(\alpha)$, because it is the ratio of quadratic to quartic polynomials.

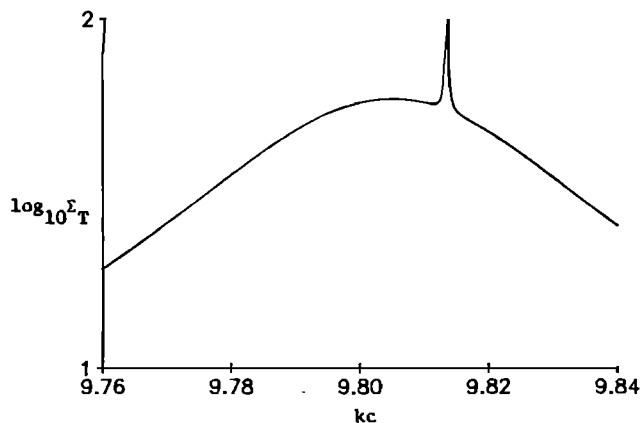


FIG. 8. Total scattering cross section of a unit amplitude plane wave near the k_{31} resonance of a unit square membrane with $c = 5$ and $\epsilon = 0.1$. The incident direction makes an angle of 5° with the membrane normal and its projection on the plane of the membrane makes an angle of 30° with a side.

We have evaluated $P_x(\alpha)$ for the eigenvalue, $k_{13}c = k_{31}c = \sqrt{10}\pi$ of the unit square membrane for several values of the sound speed ratio c , and for an incident plane wave. For the square membrane, $T_{11} = T_{22}$ for all double eigenvalues and $T_{12} = T_{21} = 0$ when $m + n$ is odd, as we can show from (62). Furthermore, when $m + n$ is odd, the response is similar to the simple eigenvalue case. Thus the lowest "true" double eigenvalue for the square is $kc = k_{13}c$. The integrals required to determine the quantities in (72) were determined numerically. The results suggest that for $c < 1$, i.e., for $c_a < c_m$, $P_x(\alpha)$ has a single maximum as in Fig. 1. However, for $c > 1$, $P_x(\alpha)$ may have either one or two maxima. The results also depend on the angle of incidence of the plane wave. A typical result is shown in Fig. 8, for k close to $k_{13} = k_{31}$. There is a relatively broad single maximum with a narrow, spiked second maximum superimposed upon the broad minimum. The condition $c > 1$ implies that the membrane material is "soft."

A uniform asymptotic expansion corresponding to the double eigenvalue can be constructed for k in an appropriate interval about k_n by employing the inner expansions (58), (69), and (70) and the outer expansions given in Sec. II. Similarly, uniform asymptotic expansions that are valid for all k can be obtained by combining inner and outer expansions taking into account the possible multiplicities of each k_n .

APPENDIX A

The scattered acoustic potential $\phi_n(\mathbf{x})$ produced by the membrane vibrating with frequency k_n and mode $\psi_n(x, y)$ is a solution of the boundary value problem:

$$\Delta\phi_n + k_n^2\phi_n = 0, \quad z > 0, \quad (\text{A1a})$$

$$\frac{\partial\phi_n}{\partial z} = 0, \quad x, y \in M, \quad (\text{A1b})$$

$$\frac{\partial\phi_n}{\partial z} = \psi_n(x, y), \quad x, y \in M, \quad (\text{A1c})$$

and the radiation condition as $r \equiv |\mathbf{x}| \rightarrow \infty$. The solution to (A1) is given by

$$\phi_n(\mathbf{x}) = G(\mathbf{x}; k_n)\psi_n, \quad (\text{A2})$$

where the integral operator G is defined in (10). In the far-field, $r \rightarrow \infty$, this expression simplifies to

$$\phi_n \sim F_n(k_n, \hat{\mathbf{r}})(e^{ik_n r}/r), \quad (\text{A3})$$

where $F_n(k_n, \hat{\mathbf{r}})$ is defined by (35) and $\hat{\mathbf{r}}$ is the unit vector in the direction of observation.

Since k_n is real, ϕ_n and $\bar{\phi}_n$ both satisfy (A1a). Hence we have

$$\nabla \cdot \bar{\phi}_n \nabla \phi_n - |\nabla \phi_n|^2 + k_n^2 |\phi_n|^2 = 0. \quad (\text{A4})$$

Integrating this expression inside the closed surface S , composed of the large hemisphere $r = R, z > 0$, and the circular region $x^2 + y^2 = R^2 (z = 0)$, and applying the divergence theorem gives us

$$\begin{aligned} R^2 \int_0^{2\pi} \int_0^{\pi/2} \phi_n \frac{\partial \bar{\phi}_n}{\partial r} \sin \phi \, d\phi \, d\theta \\ = \iiint_{r < R, z > 0} \{ |\nabla \phi_n|^2 - k_n^2 |\phi_n|^2 \} dx \, dy \, dz \\ + \iint_{r < R, z = 0} \phi_n \frac{\partial \bar{\phi}_n}{\partial z} dx \, dy. \end{aligned} \quad (\text{A5})$$

By inserting (A3) into the left side of (A5), and (A1b)–(A1c) into the second integral on the right, and then letting $R \rightarrow \infty$, we obtain

$$\begin{aligned} -ik_n \int_0^{2\pi} \int_0^{\pi/2} |F_n(k_n, \hat{\mathbf{r}})|^2 \sin \phi \, d\phi \, d\theta \\ = \iiint_{0 - \infty - \infty}^{\infty \infty \infty} \{ |\nabla \phi_n|^2 - k_n^2 |\phi_n|^2 \} dx \, dy \, dz \\ + \iint_M \phi_n(x, y, 0, k_n) \psi_n(x, y) dx \, dy. \end{aligned} \quad (\text{A6})$$

Finally, we substitute (A2) evaluated at $z = 0$ for $\phi_n(x, y, 0, k_n)$ into (A6) and recall the definition of a_n , given by (27), to arrive at (40) and (41).

We now prove a similar result which will be used in Appendix C. Let ψ_n^1 and ψ_n^2 be two eigenfunctions of the membrane corresponding to the same eigenfrequency $k_n c$. Let ϕ_n^1 and ϕ_n^2 be the corresponding solutions of (A1) with ψ_n replaced by ψ_n^1 and ψ_n^2 , respectively. It then follows from (A1a) that

$$\nabla \cdot (\phi_n^1 \nabla \bar{\phi}_n^2 - \bar{\phi}_n^2 \nabla \phi_n^1) = 0. \quad (\text{A7})$$

We first integrate this expression over the interior of S and then let $R \rightarrow \infty$ to obtain

$$\text{Im} \langle \psi_n^1, G(k_n) \psi_n^2 \rangle = -k_n \int_0^{2\pi} \int_0^{\pi/2} F_n^1 \bar{F}_n^2 \sin \phi \, d\phi \, d\theta, \quad (\text{A8})$$

where the F_n^i are defined by (35) with ψ_n replaced by ψ_n^i .

APPENDIX B

To show that $R > 0$, where R is defined in (41), we first employ the plane-wave expansion of $g(x, y, 0 | \xi, \eta, 0; k_n)$. It is given by

$$g(x, y, 0 | \xi, \eta, 0; k_n) = \frac{-i}{2\pi} \iint_{-\infty}^{\infty} \frac{\exp\{i[\sigma(x - \xi) + \tau(y - \eta)]k_n\}}{(k_n^2 - \sigma^2 - \tau^2)^{1/2}} d\sigma d\tau, \quad (B1)$$

where the branch of the square root function is defined by $\sqrt{-1} \equiv i$. When (B1) is inserted into the definition of a_n given in (27) and the limits of integration are interchanged, we obtain

$$a_n = \frac{i}{2\pi} \iint_{-\infty}^{\infty} \frac{|\hat{\psi}_n|^2 d\sigma d\tau}{(k_n^2 - \sigma^2 - \tau^2)^{1/2}}. \quad (B2)$$

In (B2) $\hat{\psi}_n$ is the finite Fourier transform of ψ_n defined by

$$\hat{\psi}_n(\sigma, \tau) = \iint_M \psi_n(x, y) \exp[i(\sigma x + \tau y)k_n] d\sigma d\tau. \quad (B3)$$

It follows directly from (B2) that

$$a_n = \frac{i}{2\pi} \iint_{\sigma^2 + \tau^2 < k_n^2} \frac{|\hat{\psi}_n|^2 d\sigma d\tau}{[k_n^2 - (\sigma^2 + \tau^2)]^{1/2}} + \frac{1}{2\pi} \iint_{\sigma^2 + \tau^2 > k_n^2} \frac{|\hat{\psi}_n|^2 d\sigma d\tau}{(\sigma^2 + \tau^2 - k_n^2)^{1/2}}, \quad (B4)$$

and have $R = \text{Re}(a_n)$ is positive.

APPENDIX C

The quantity $\Delta(\alpha)$, which we now show does not vanish, is the determinant of the system (60). That is

$$\Delta(\alpha) = \det(T + 2\alpha I), \quad (C1)$$

where I is the 2×2 identity matrix and T is the 2×2 matrix with components T_{ij} defined by (62). Since α is real, it follows from (C1) that $\Delta(\alpha) \neq 0$ if and only if T has no real eigenvalues.

To demonstrate that T has no real eigenvalues we first denote any eigenvalue of T and a corresponding unit eigenvector by μ and $\mathbf{x} = (x_1, x_2)$, respectively. It then follows from the Rayleigh quotient for μ that

$$\mu = T_{11}|x_1|^2 + T_{22}|x_2|^2 + 2T_{12} \text{Re}(x_1 \bar{x}_2). \quad (C2)$$

Rewriting x_l in polar form, $x_l = r_l e^{i\theta_l}$, $l = 1, 2$ and inserting this into (C2), we obtain

$$\text{Im } \mu = b_1 \left(r_1 + \frac{b_3 r_2 \cos \delta}{b_1} \right)^2 + b_2 \left(1 - \frac{(b_3 \cos \delta)^2}{b_1 b_2} \right) r_2^2. \quad (C3)$$

In (C3) we have used the following notation: $b_1 = \text{Im}(T_{11})$, $b_2 = \text{Im}(T_{22})$, $b_3 = \text{Im}(T_{12})$, and $\delta = \theta_1 - \theta_2$. However, it follows from the definition (62) of T_{ij} and from the analysis of Appendix A [see (A6)], that

$$b_l = -k_n \int_0^{\pi/2} \int_0^{2\pi} |F_n^l(k_n, \hat{\mathbf{r}})|^2 \sin \phi d\phi d\theta, \quad l = 1, 2. \quad (C4)$$

Here, F_n^l is the directivity factor given by (35) with ψ_n replaced by ψ_n^l . Thus b_1 and b_2 are both negative. From (A8) we have

$$b_3 = k_n \int_0^{\pi/2} \int_0^{2\pi} F_n^1 \bar{F}_n^2 \sin \phi d\phi d\theta, \quad (C5)$$

which is a real quantity. The Cauchy-Schwartz inequality applied to (C5) gives

$$|b_3|^2 \leq |b_1| |b_2| \equiv b_1 b_2 \quad (C6)$$

from which we deduce that the factor multiplying $b_2 r_2^2$ in (C3) is positive. This implies that $\text{Im } \mu < 0$ and hence T has no real eigenvalues.

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