III. LOVE WAVES

The discussion of Love waves in Sec. V of the subject paper contains several conceptual and typographical mistakes. The corrected equations are given below. The equation of motion should be

$$\frac{\partial s_{12}}{\partial x}+\frac{\partial s_{23}}{\partial z}-\eta P\frac{\partial \omega}{\partial x}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{14}$$

where stresses s_{12} and s_{23} should be

$$s_{12}=\eta\,\mu\frac{\partial v}{\partial x},\quad s_{23}=\eta\,\mu\frac{\partial v}{\partial z}. \tag{15}$$

We then have

$$(1-I_j)\frac{\partial^2 v_j}{\partial x^2}+\frac{\partial^2 v_j}{\partial z^2}=\frac{1}{\beta_j^2}\frac{\partial^2 v_j}{\partial t^2}, \tag{16}$$

with

$$I_j=P/2\mu_j,\quad \beta_j^2=\eta\,\mu_j/\rho_j,\quad j=1,2. \tag{17}$$

The expressions for m^2 and r^2 should have been

$$m^2=(c^2/\beta_1^2-1+I_1)k^2, \tag{18}$$

$$r^2=(1-I_2-c^2/\beta_2^2)k^2. \tag{19}$$

The dispersion equation should be modified to

$$\tan\left[\left(\frac{c^2}{\beta_1^2}-1+I_1\right)^{1/2}kH\right]=\frac{\mu_2(1-I_2-c^2/\beta_2^2)^{1/2}}{\mu_1(c^2/\beta_1^2-1+I_1)^{1/2}}.$$

[1]A. Chattopadhyay, S. Bose, and M. Chakraborty, "Reflection of elastic waves under initial stress at a free surface: P and SV motion," J. Acoust. Soc. Am. 72, 255–263 (1982).

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Propagation of plane waves in a pre-stressed elastic medium

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A homogeneous, isotropic elastic medium having initial axial stresses in two orthogonal directions is considered. The dynamic equations for superimposed small deformations are stated. The propagation of plane waves in such a medium is discussed. It is shown that pure longitudinal and shear waves can propagate only in certain specific directions which are defined. The present analysis corrects a fundamental error in a recent paper in this Journal [J. Acoust. Soc. Am. 72, 255–263 (1982)].

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INTRODUCTION

The problem of small amplitude waves propagating in pre-stressed or pre-strained elastic media has been discussed in detail by several authors. Two quite different presentations of this topic, in Cartesian and tensorial rotation, may be found in Refs. 1 and 2, respectively. Wave propagation in a pre-stressed homogeneous, isotropic, linearly elastic medium has been considered in a recent paper in this *Journal*.[3] However, the authors assume a form of solution which does not satisfy their equations of motion. Reference 3 is concerned with the reflection of plane waves from a free surface in a half-space. Exactly the same error occurs in two very similar papers.[4,5] As a result, the numerical findings and conclusions of Refs. 3–5 are in doubt.

I. EQUATIONS OF MOTION

We consider a state of uniform initial stress with principal stresses S_{11} and S_{22} along the x and y axes, respectively, and $S_{12}=0$. The material is in plane strain with orthotropic symmetry. The principle directions of symmetry are the coordinate axes. The equations of motion for superimposed deformations are given by Biot [Ref. 1, Chap. 5, Eq. (4.5)]. Let u and v be the displacements in the x and y directions, then

$$\begin{aligned}&B_{11}\frac{\partial^2 u}{\partial x^2}+\left(Q+\frac{P}{2}\right)\frac{\partial^2 u}{\partial y^2}+\left(B_{12}+Q-\frac{P}{2}\right)\frac{\partial^2 v}{\partial x\partial y}\\&\qquad=\rho\frac{\partial^2 u}{\partial t^2}\\&B_{22}\frac{\partial^2 v}{\partial y^2}+\left(Q-\frac{P}{2}\right)\frac{\partial^2 v}{\partial x^2}+\left(B_{12}+Q-\frac{P}{2}\right)\frac{\partial^2 u}{\partial x\partial y}\\&\qquad=\rho\frac{\partial^2 v}{\partial t^2}.\end{aligned} \tag{1}$$

Here $P=S_{22}-S_{11}$, and the elastic constants Q,B_{11},B_{22},B_{12} are also functions of the initial stress. Only Q may be expressed explicitly in terms of the finite initial stresses S_{11}, S_{22} and initial strains λ_1, λ_2 [Ref. 1, Chap. 2, Eq. (7.15)],

$$2Q=\begin{cases}P(\lambda_1^2+\lambda_2^2)/(\lambda_2^2-\lambda_1^2), & S_{11}\neq S_{22}\\ \\ B_{11}-B_{12}, & S_{11}=S_{22}.\end{cases} \tag{2}$$

However, in the limit as S_{11} and S_{22} tend to zero, the elastic constants reduce to their isotropic, stress free limits: $B_{11}=B_{22}=\lambda+2\mu$, $B_{12}=\lambda$ and $Q=\mu$, where λ,μ are the Lamé constants.

The elastic constants are defined through the stress–strain relations

$$\begin{aligned} s_{11}&=B_{11}e_{xx}+B_{12}e_{yy},\\ s_{22}&=B_{21}e_{xx}+B_{22}e_{yy},\\ s_{12}&=2Qe_{xy},\end{aligned} \tag{3}$$

where $B_{21}=B_{12}-P$ (Ref. 1, Chap. 2, Sec. 8). In the paper by Chattopadhyay *et al.*[3] the authors assume an isotropic stress–strain relation. Their values of the elastic constants are given by Eq. (2) of Ref. 3.

II. PROPAGATION OF PLANE WAVES

Consider the plane wave

$$\mathbf{u}=A\mathbf{p}e^{ik(\mathbf{n}\cdot\mathbf{x}-ct)},\quad |\mathbf{n}|=1. \tag{4}$$

Here k is the wavenumber, c the phase speed, $\mathbf{n}$ the direction of propagation, and $\mathbf{p}$ the displacement vector. For a given wavenumber and propagation direction, c and $\mathbf{p}$ are determined by Eq. (1). Substituting for $\mathbf{u}$ gives the characteristic equation:

$$\begin{bmatrix} B_{11}n_1^2+(Q+P/2)n_2^2-\rho c^2, & (B_{12}+Q-p/2)n_1n_2\\ \\ (B_{12}+Q-P/2)n_1n_2, & (Q-P/2)n_1^2+B_{22}n_2^2-\rho c^2\end{bmatrix}\times\begin{bmatrix}p_1\\ \\ p_2\end{bmatrix}=\begin{bmatrix}0\\ \\ 0\end{bmatrix}. \tag{5}$$

The secular equation which determines $c=c(\mathbf{n})$ is found by setting the determinant of the matrix to zero. The resulting quadratic in c^2 can be factored into two sheets: a quasilongitudinal and a quasishear. We note that Eq. (5) is not equivalent to the characteristic equation of a linearly elastic anisotropic solid. This is evident by noting that the Kelvin–Christoffel stiffnesses of Eq. (5) do not satisfy the required symmetries (Ref. 6, Sec. 7.2). This is because the incremental stresses and strains do not have an associated strain-energy function.[1]

In his book, Biot[1] shows that pure longitudinal and shear waves can propagate in the x and y directions. It is of interest to see if pure modes propagate in any other directions. Such directions are known as specific directions. Longitudinal specific directions are found by putting $n_1=p_1=\cos\theta$ and $n_2=p_2=\sin\theta$ in Eq. (5). We find that $\theta=0$, $\pi/2$, and θ_0, where

$$\tan\theta_0=\left(\frac{B_{11}-B_{21}-2Q}{B_{22}-B_{12}-2Q}\right)^{1/2}. \tag{6}$$

If the right-hand side of Eq. (6) is real, then θ_0 is a specific direction, with longitudinal wave speed

$$\rho c^2=B_{11}-(B_{11}-B_{12}-2Q)\sin^2\theta_0. \tag{7}$$

Putting $n_1=-p_2=\cos\theta$, $n_2=p_1=\sin\theta$ in Eq. (5) determines shear specific directions. These are again $\theta=0$, $\pi/2$, and θ_0, with the shear wave speed in direction θ_0 given by

$$\rho c^2=Q+P+(B_{11}-B_{12}-2Q)\cos^2\theta_0. \tag{8}$$

When the initial stresses are equal, we have $P=0$, $B_{11}=B_{22}$, and $\theta_0=\pi/4$. The longitudinal and shear speeds in the specific directions are $(B_{11}/\rho)^{1/2}$ and $(Q/\rho)^{1/2}$, respectively. Away from these directions the speeds vary as in an orthotropic solid with elastic stiffnesses $C_{11}=C_{22}=B_{11}$, $C_{12}=B_{12}$, and $C_{66}=Q$, see Ref. 6, Sec. 9.3.

Since the wave speeds depend upon the propagation direction $\mathbf{n}$, the group velocity of the wave (4) is not, in general, equal to the phase velocity $c\mathbf{n}$. This is a common phenomenon in anisotropy.[6] The group velocity is important since it is the velocity with which the energy is propagated. Let $\mathbf{n}=(\cos\theta,\sin\theta)$, then from Eq. (5) we have $c=c(\theta)$. The group velocity (wave velocity) is $c_g\mathbf{m}$, $|\mathbf{m}|=1$, where

$$c_g\,\mathbf{m}=c\left(\mathbf{n}+\frac{\partial c}{\partial\theta}c^{-1}\mathbf{e}_\theta\right). \tag{9}$$

Here $\mathbf{e}_\theta=(-\sin\theta,\cos\theta)$. The phase and group velocities are equal when $\partial c/\partial\theta=0$. This occurs at $\theta=0$, $\pi/2$ but not at θ_0, in general.

Finally, we note that the analysis of Sec. 1, Ref. 3 concerning the form of traveling wave solutions is totally incorrect. The relevant equations to be solved are in Eq. (4) of Ref. 3. The method of solution would be to consider a wave field as in Eq. (4) above and determine the resulting characteristic equation. However, the authors assumed an *isotropic* solution defined by two potential functions. Now, the use of potential functions in elastodynamics is not possible under general anisotropy, but only in certain special cases, e.g., isotropy and transverse isotropy. The errors of their way are obvious upon noting that their final solution [Ref. 3, Eq. (10)] does not satisfy the Eq. (4) of Ref. 3.

Note added in proof: The topic of the present letter is discussed in greater detail in a recent article by I. Tolstoy, "On elastic waves in prestressed solids," J. Geophys. Res. **87** (B8), 6823–6827 (1982).

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