Back reflection of ultrasonic waves from a liquid–solid interface

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A new acoustic phenomenon has recently been observed in experiments where a bounded beam of ultrasound is incident upon a smooth liquid–solid interface. A significant amount of coherent radiation is found to be backscattered in the general direction of incidence. The angle of back reflection is observed to be equal to the critical Rayleigh angle or leaky wave angle. Most of these observations were made during experiments on the Schoch displacement effect, and therefore it has been tacitly assumed that the back reflection is strongly dependent upon the angle of incidence, as is the case for the beam shifting in the Schoch effect. We present a theoretical basis for this new phenomenon. A two-dimensional incident beam of Gaussian profile is considered. By a careful analysis we isolate that part of the field on the interface which has Fourier components corresponding to backward propagating waves in the liquid. This subset of the total wave field is then considered separately and it is shown to display a maximum in a certain direction, close to the critical Rayleigh angle. This peak in the angular pattern of the scattered field corresponds to an evanescent reflection boundary. We discuss the dependence of the effect upon certain parameters. The amplitude is shown to decrease as the beam width is increased, and it increases with increasing Schoch displacement. This backscattering is present for all angles of incidence; there is nothing inherently special about the Rayleigh angle.

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INTRODUCTION

Recently a new phenomenon in acoustics has been reported,[1–5] though as yet no physical theory has been presented to support these observations. The phenomenon is the backscattering of a beam of ultrasound from a fluid–solid interface. The incident beam originates from an acoustic transducer in the fluid. At the interface the beam is specularly reflected back into the liquid and transmitted into the solid. In addition, when the incident angle is at the critical angle, an effect known as the Schoch displacement occurs. This involves the lateral displacement of the reflected beam, and its origin is well understood.[6] Experimenters who were primarily interested in observing the Schoch displacement also noticed that a significant amount of energy is radiated from the interface back in the direction of incidence. A Schlieren photograph of such radiation may be seen in Ref. 3.

In this paper we explain this phenomenon in terms of leaky waves propagating along the interface in the backward direction. Our approach is to model the reflection coefficient by a simplified form which still exhibits the poles corresponding to leaky waves in either direction along the interface. We consider a two-dimensional geometry with an incident beam of Gaussian profile. In practice, the transducers used in experiment produce almost perfectly Gaussian beams. Under these assumptions, we are able to calculate explicitly the field on the interface. The scattered field in the fluid can then be calculated using a representation integral. However, this latter step is simplified by noting that only a certain part of the interfacial field contributes to backward propagating waves. This field, considered in isolation, is shown to produce a backward propagating beam effect near the critical angle.

Previously, it has been thought that the backscattering depends critically upon the angle of incidence.[4] This may be due to some confusion of the effect with that of Schoch displacement, in which the angle of incidence is critical. We show that the angle of incidence is not important, but the radiated energy will only be observed at or near the critical angle, which we call the leaky wave angle θ_l. Our results indicate that the backscattering is due to a leaky wave reflection zone. This reflection zone is a new phenomenon and complements previous descriptions of evanescent shadow and transmission zones occurring in acoustics and electromagnetics.[7–10] We find that the back reflection depends mainly upon two parameters. The first is the dimensionless beamwidth (kw_0), where k is the incident wavenumber and w_0 is the half-width of the beam on the interface. The amplitude of the back reflection decreases exponentially as $(kw_0)^2$ increases. The second critical parameter is the dimensionless Schoch displacement $(k\Delta s)$, where Δs is the Schoch displacement, known to be inversely proportional to the imaginary part of the leaky wavenumber.[6] We find that the back reflection increases as $(k\Delta s)$ increases. The above two findings are in agreement with experimental observation.[4]

I. THE REFLECTION COEFFICIENT

Before considering bounded beams of waves, it is useful to consider the interaction of a single plane wave with the interface. The general problem for a beam or for an arbitrary source distribution may then be solved by a Fourier superposition.

Let the plane wave be incident from the liquid half space $z < 0$, with the liquid–solid interface in the plane $z = 0$. The displacement field within the liquid (assumed to be ideal) may be written in terms of a potential ϕ, which is proportional to the pressure field. Let the incident partial wave $\phi_{\rm inc}$ be

$$\phi_{\rm inc} = \exp i(k_x x + k_z z), \quad (1)$$

where

$$k_z \equiv (k^2 - k_x^2)^{1/2} \quad (2)$$

and k is the wavenumber of the incident wave. The time harmonic factor $e^{-i\omega t}$ is suppressed throughout. One reflected wave and two transmitted waves are generated at the interface. The reflected wave in the liquid is

$$\phi_{\rm ref} = R(k_x) \exp i(k_x x - k_z z), \quad (3)$$

where the reflection coefficient is[11]

$$R(k_x) = \frac{F(k_x) - i\rho k_T^4 (k_x^2 - k_L^2)^{1/2}/k_z}{F(k_x) + i\rho k_T^4 (k_x^2 - k_L^2)^{1/2}/k_z}. \quad (4)$$

In this expression, k_L and k_T are the wavenumbers of longitudinal and transverse waves in the solid, ρ is the ratio of liquid to solid mass densities, and $F(k_x)$ is the Rayleigh function for the solid,

$$F(k_x) = (2k_x - k_T^2)^2 - 4k_x^2 (k_x^2 - k_L^2)^{1/2} (k_x^2 - k_T^2)^{1/2}. \quad (5)$$

The two transmitted longitudinal and transverse waves propagate in directions determined by Snell's law, with amplitudes which follow from the interfacial boundary conditions.[1] Since we are only interested in phenomena observed in the liquid, we shall not discuss the transmitted waves further.

The interface can support boundary waves which decay with distance from the interface. These correspond to poles of the reflection coefficient $R(k_x)$ in the complex k_x plane. Since R is a symmetric function of k_x, it follows that the poles occur in pairs symmetric about the origin. In the lossless case, i.e., k, k_L, and k_T all real, it may be shown[12] that a pole occurs for k_x greater than either k or k_T (note, $k_L < k_T$). The corresponding boundary wave, called the Scholte wave, does not decay as it propagates along the interface and has a wave speed lower than any other in the problem. For most interfaces, we have $k > k_R$, where k_R is the Rayleigh wavenumber for the solid, in which case a complex pole exists near k_R with positive imaginary part. The so-called leaky Rayleigh wave attenuates with distance along the interface while shedding its energy into the liquid.

An inhomogeneous wave in the liquid generates these different boundary waves when it strikes the interface. However, only the leaky Rayleigh wave is capable of leaking energy back into the liquid. We conclude that the phenomena of beam displacement and backscattering are due mainly to the leaky wave modes. Accordingly, we seek an approximation to $R(k_x)$ which exhibits the leaky wave poles. Bertoni and Tamir[6] modeled the reflection coefficient by isolating the pole responsible for the forward propagating leaky waves. We are also interested in the effect of the backward leaky waves, therefore we assume the following simplified form for R:

$$R(k_x) \sim R_0(k_x) \equiv (k_x^2 - k_0^2)/(k_x^2 - k_p^2). \quad (6)$$

Here k_p is the complex leaky wave pole of $R(k_x)$ and k_0 is the complex zero of $R(k_x)$ which occurs near k_p. In most instances,[12] it turns out that k_p is very close in value to the Rayleigh wavenumber k_R,

$$k_p = k_R(1 + \epsilon_2 + i\epsilon_1), \quad (7)$$

where $0 < \epsilon_1 \ll 1$ and $\epsilon_2 = O(\epsilon_1^2)$. Particular values of ϵ_1 for typical interfaces are given in Refs. 6 and 12. When there are no losses present, we have $k_0 = k_p^*$, the complex conjugate. If losses are present in the media, $k_0 \neq k_p^*$ (Ref. 12) and the imaginary part of k_p is most sensitive to variations in the imaginary part of the transverse wavenumber k_T.

The model reflection coefficient $R_0(k_x)$ does not have poles corresponding to the Scholte surface waves. However, as mentioned before, these waves cannot provide the mechanism for the phenomena considered. Even in the lossy case, the Scholte wave becomes leaky in nature but the energy radiated back into the liquid is noncoherent. This follows from the fact that the Scholte wave speed is less than the speed of sound in the liquid. Our approximate reflection coefficient does not display the branch points of $R(k_x)$. However, as justification for its use we note from Eqs. (4), (6), and (7) that $R_0(k_x)$ has the correct analytic behavior near the poles at $\pm k_p$. We mention that Bertoni and Tamir's approximate reflection coefficient[6] is obtained from Eq. (6) by removing the squares. It has been shown[13] that their theory gives almost exactly the same results for the forward displacement whether the exact or approximate reflection coefficient is used.

II. THE INCIDENT BEAM

In order to demonstrate analytically the reflection and backscatter of ultrasonic beams it is necessary to have a particular beam profile. The most common one used in the relevant experiments[14] is the Gaussian beam, which may be produced by subjecting the transducer face to a Gaussian pressure distribution. Consider a two-dimensional transducer whose face is centered at the vector position $\mathbf{x}_S = (-X_0, -h)$, where $X_0 = h \tan \theta_i$ and θ_i is the angle of incidence of the beam at the interface $z = 0$ (see Fig. 1). The transducer face is parallel to the unit vector $\mathbf{e}_T$ and the beam direction is that of the unit vector $\mathbf{e}_B$. One way of expressing a Gaussian beam is in terms of a point source in complex space.[15] Accordingly, we write the incident field as

$$\phi_{\rm inc}(\mathbf{x}) = \tfrac{1}{4} i H_0^{(1)}(kD), \quad (8)$$

where $H_0^{(1)}$ is the Hankel function of the first kind, and the complex distance D is

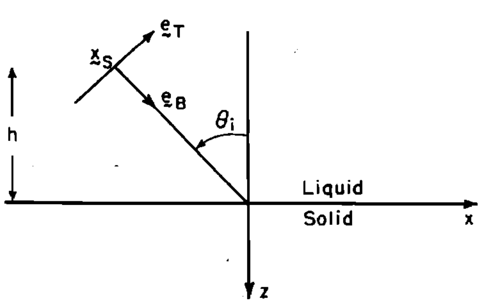

FIG. 1. Two-dimensional geometry showing the angle of incidence θ_i and the beam direction $\mathbf{e}_B$.

$$D \equiv |\mathbf{x} - (\mathbf{x}_S + ib\mathbf{e}_B)|. \tag{9}$$

Here b is the length characterizing the beam. The branch points of D lie at $\mathbf{x}_S \pm b\mathbf{e}_T$ in real space. Therefore, the field of Eq. (8) is an exact solution of the equations of motion in the liquid outside the circle of radius b centered at $\mathbf{x}_S$.

Let an arbitrary point $\mathbf{x}$ be parameterized by distances along the beam axis and normal to it, thus

$$\mathbf{x} = \mathbf{x}_S + X\mathbf{e}_T + Z\mathbf{e}_B. \tag{10}$$

Near the beam axis, such that $X/b \ll 1$, we may expand D as

$$D = Z - ib + iX^2/2(b + iZ) + \cdots. \tag{11}$$

Assuming that

$$k|Z - ib| \gg 1, \tag{12}$$

we can use the asymptotic form for the Hankel function of large argument to obtain

$$\phi_{\text{inc}} = \frac{e^{i\pi/4}}{(8\pi)^{1/2}} \frac{\exp[k(b + iZ)]}{\sqrt{k}(b + iZ)} \exp\left(\frac{-kX^2}{2(b + iZ)}\right) \tag{13}$$

on the transducer face $Z = 0$, and Eq. (13) gives a Gaussian profile of half-width w_t, where

$$w_t = (2b/k)^{1/2}. \tag{14}$$

Away from the transducer, the beam spreads out but remains Gaussian. At a distance Z along the beam axis, the beam half-width is w, where

$$w = w_t(1 + Z^2/b^2)^{1/2}. \tag{15}$$

If the complex point source solution is to model the actual transducer profile faithfully, then the field near the transducer edges must be small as compared to at the center in order to avoid diffraction effects. This implies that a, the transducer half-width, is of the same order of magnitude as w_t. Therefore, by Eqs. (12) and (14) we have the necessary condition that

$$(ka)^2 = O(kb) \tag{16}$$

from which it follows that $kb \gg ka \gg 1$. Finally we note that Eq. (13) may also be derived using the parabolic approximation to the wave equation.[15]

III. THE FIELD ON THE INTERFACE

The incident field at the interface follows from Eq. (13) by substituting for the beam coordinates X, Z in terms of the fixed coordinates x, z with $z = 0$. The term

$$[k(b + iZ)]^{1/2} \exp[-kX^2/2(b + iZ)]$$

does not depend strongly upon Z on the interface, so we replace Z by $h \sec\theta_i$, its value at the intersection of the beam axis with the interface. Thus, we have

$$\phi_{\text{inc}}(x,0) = A_0 \exp(ik \sin\theta_i x)\exp(-kx^2\cos^2\theta_i/2b_0), \tag{17}$$

where

$$A_0 = e^{i\pi/4}e^{kb_0}/(8\pi k b_0)^{1/2}, \tag{18}$$

and the complex distance b_0 is

$$b_0 = b + ih\sec\theta_i. \tag{19}$$

The Fourier transform of a function $f(x)$ is defined as $\bar{f}(k_x)$ thus:

$$\bar{f}(k_x) \equiv \int_{-\infty}^{\infty} e^{-ik_x x} f(x)\, dx. \tag{20}$$

From Eq. (17) we obtain

$$\bar{\phi}_{\text{inc}}(k_x,0) = (e^{i\pi/4}/2k)\sec\theta_i e^{kb_0} \times \exp[-b_0\sec^2\theta_i(k_x - k_i)^2/2k], \tag{21}$$

where

$$k_i = k\sin\theta_i. \tag{22}$$

The scattered field ϕ in the liquid half-space $z<0$ is got by multiplying $\bar{\phi}_{\text{inc}}$ by the canonical reflection coefficient R_0 of Eq. (6):

$$\phi(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_0(k_x)\bar{\phi}_{\text{inc}}(k_x,0)e^{i(k_x x - k_z z)}\, dk_x, \tag{23}$$

where k_z was defined in Eq. (2).

Let us first consider the field on the interface $z = 0$. The integration in Eq. (23) can then be achieved by rewriting $R_0(k_x)$ as

$$R_0(k_x) = R_0(k_i) + R_1(k_x), \tag{24}$$

with

$$R_1(k_x) = \left(\frac{k_p^2 - k_0^2}{k_p^2 - k_i^2}\right)\left[1 + \frac{(k_p^2 - k_i^2)}{2k_p} \times \left(\frac{1}{k_x - k_p} - \frac{1}{k_x + k_p}\right)\right]. \tag{25}$$

The constant terms in $R_0(k_x)$ reproduce the incident field $\phi_{\text{inc}}(x,0)$ while the singular terms can be handled using the result[16]

$$\int_{-\infty}^{\infty} [\exp(-\xi^2 + i2\alpha\xi)/(\xi - \beta)]d\xi = si\pi e^{-\alpha^2}e^{(\alpha + i\beta)^2}\text{erfc}[-s(\alpha + i\beta)], \tag{26}$$

where $s = \text{sgn}[\text{Im}(\beta)]$ and erfc ( ) is the complementary error function. We find that

$$\phi(x,0) = \phi_0(x,0) + \phi_L(x,0), \tag{27}$$

where ϕ_0 is the specularly reflected beam which would be predicted by geometrical optics, while ϕ_L contains the leaky wave effects. Thus

$$\phi_0(x,0) = \phi_{\text{inc}}(x,0)R_0(k_i) \tag{28}$$

and

$$\phi_L(x,0) = \phi_{\text{inc}}(x,0)\left[\frac{k_p^2 - k_0^2}{k_p^2 - k_i^2} + \left(\frac{k_p^2 - k_0^2}{4k_p}\right)i\pi^{1/2}w_0 \times \left[e^{\lambda_+^2}\text{erfc}(\lambda_+) + e^{\lambda_-^2}\text{erfc}(\lambda_-)\right]\right], \tag{29}$$

where

$$w_0 = (2b_0/k)^{1/2}\sec\theta_i \tag{30}$$

and

$$\lambda_\pm = -\frac{i}{2}w_0k_p \pm \left(\frac{i}{2}w_0k_i - \frac{x}{w_0}\right). \tag{31}$$

The magnitude of the complex distance w_0 is the half-width of the incident beam profile projected on the interface. Usually the distance h is small as compared with b [see Eq. (19)], and we may take w_0 as real.

Let us split ϕ_L as follows:

$$\phi_L(x,0)=\phi_1(x,0)+\phi_2(x,0), \tag{32}$$

where

$$\phi_2(x,0)\equiv\phi_{\text{inc}}(x,0)\left[(k_p^2-k_0^2)/4k_p\right] \times i\pi^{1/2}w_0 e^{\lambda_-^2}\,\mathrm{erfc}(\lambda_-). \tag{33}$$

The field ϕ_1, which is analogous but not identical to the field v_1 of Bertoni and Tamir,[6] contains the forward propagating leaky waves, and can be shown to be the major cause of the Schoch displacement.[6] The backward propagating leaky waves are contained in ϕ_2. Define the real and imaginary parts of k_p thus,

$$k_p=k_l+i\epsilon,\quad k_l,\epsilon>0. \tag{34}$$

The imaginary part ϵ has been shown to be related to the Schoch displacement Δs by[6]

$$\Delta s=2/\epsilon. \tag{35}$$

If the angle of incidence is such that $k_i=k_l$, then $\phi(x,0)$ has a simple representation in terms of the three parameters

$$\alpha=x/w_0,\quad \beta=w_0/\Delta s,\quad \gamma=k_l w_0. \tag{36}$$

We obtain

$$\phi(w_0\alpha,0)=A_0e^{i\alpha\gamma-\alpha^2}\left[1-\left(\frac{\pi^{1/2}\beta\gamma 2}{\gamma+2i\beta}\right)\left[e^{(\alpha-\beta)^2}\mathrm{erfc}(\beta-\alpha) + e^{(\alpha+\beta-pi\gamma)^2}\mathrm{erfc}(\alpha+\beta-i\gamma)\right]\right]. \tag{37}$$

The final term in this expression represents ϕ_2. Numerical experiments indicate that the relative magnitude of ϕ_2 increases as β increases and γ decreases. Some examples for several values of β and γ are shown in Fig. 2. It can be shown quite simply that as $\gamma\to\infty$, ϕ_2 becomes of negligible importance and ϕ_0 and ϕ_1 reduce to the quantities v_0 and v_1 of Bertoni and Tamir,[6] apart from a multiplicative factor. Let us now consider the contribution to ϕ_2 from backward propagating leaky waves.

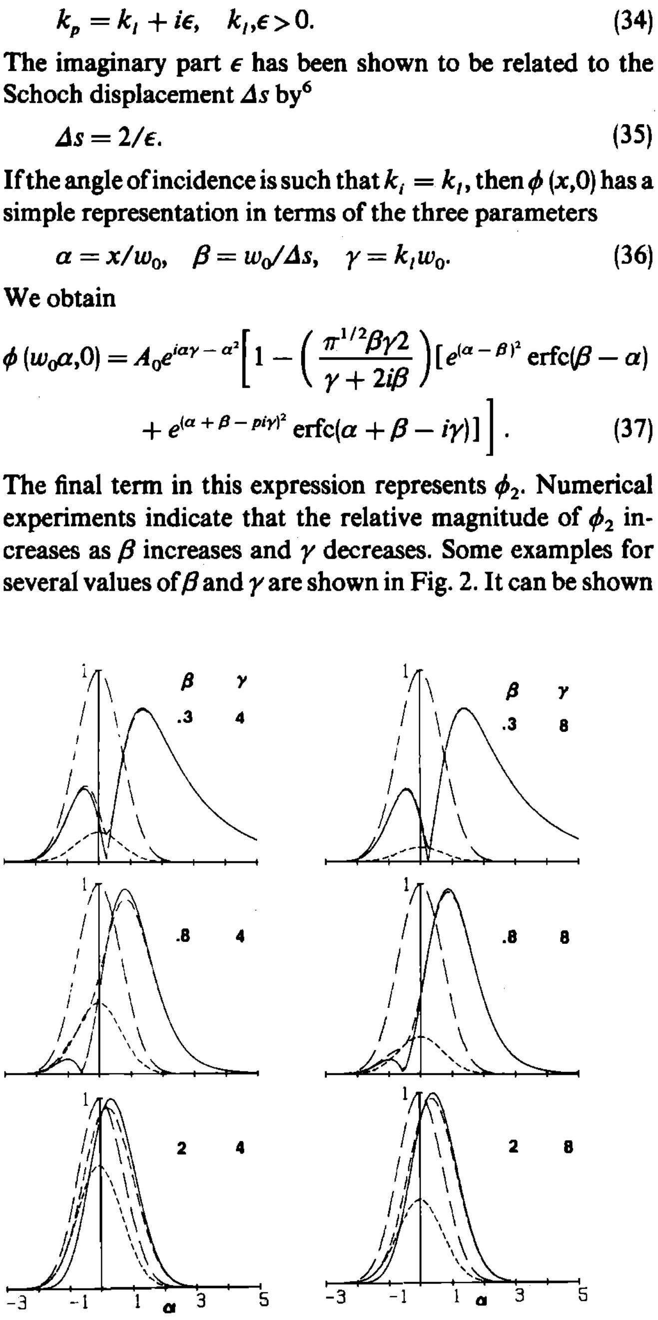

FIG. 2. The incident field and the separate parts of the total field on the interface for several values of β and γ: ϕ_{inc} — —; $|\phi_0+\phi_1|$— — —; $|\phi_2|$----; $|\phi|$—.

We first note the result for the complementary error function of complex argument,[17] that

$$\mathrm{erfc}(x-iy)=\mathrm{erfc}(x)+\frac{e^{-x^2}}{\pi}\sum_{-\infty}^{\infty}\frac{e^{-n^2/4}}{n^2+4x^2}\{-2x+e^{i2xy} \times[2x\,\mathrm{ch}(ny)+in\,\mathrm{sh}(ny)]\}+\text{error}. \tag{38}$$

This formula is exact if $y=0$, otherwise there is a relative error of the order 10^{-17}. Henceforth we will neglect this error and take the right-hand side of Eq. (38) to be an analytic representation of erfc $(x-iy)$.

For an arbitrary angle of incidence, we have by its definition in Eq. (31), that

$$\lambda_-=p-iq, \tag{39}$$

where

$$p=x/w_0+w_0/\Delta s, \tag{40}$$

$$q=w_0(k_i+k_l)/2. \tag{41}$$

The quantity p is a dimensionless shifted coordinate. We note that its zero is at $x=-w_o^2/\Delta s$. Combining Eqs. (33), (38), and (39), we have

$$\phi_2(x,0)=e^{-ik_pw_0p}\psi_B^0(p)+e^{ik_\epsilon w_0 p}\psi_F^0(p), \tag{42}$$

where

$$\psi_B^0=\frac{1}{2}A_B\left(\mathrm{erfc}(p)-\frac{2p}{\pi}e^{-p^2}\sum_{-\infty}^{\infty}\frac{e^{-n^2/4}}{n^2+4p^2}\right), \tag{43}$$

$$\psi_F^0=A_B\frac{e^{-p^2}}{2\pi}\sum_{-\infty}^{\infty}\frac{e^{-n^2/4}}{n^2+4p^2}\left[2p\,\mathrm{ch}(nq)+in\,\mathrm{sh}(nq)\right], \tag{44}$$

$$k_\epsilon=k_i-i\epsilon, \tag{45}$$

$$A_B=A_0 i\pi^{1/2}w_0\left[(k_p^2-k_0^2)/2k_p\right] \times\exp\left[-q^2+(w_0/\Delta s)^2(1-ik_i\Delta s)\right], \tag{46}$$

and A_0 is defined in Eq. (18). The form of Eq. (42) suggests that $\phi_2(x,0)$ is composed of a backward propagating leaky wave with amplitude modulation $\psi_B^0(p)$ and a wave similar to the incident field with modulation ψ_F^0. However, the form of Eq. (42) is deceptive since both ψ_B^0 and ψ_F^0 are singular at $p=0$. To overcome this problem, we remove the singularity from ψ_F^0, and write

$$\phi_2(x,0)=e^{-ik_pw_0p}\psi_B(p)+e^{ik_\epsilon w_0p}\psi_F(p) \equiv\phi_B(p)+\phi_F(p), \tag{47}$$

where now

$$\psi_B=\psi_B^0+e^{i2pq}A_B(4\pi p)^{-1}e^{-p^2}, \tag{48}$$

$$\psi_F=\psi_F^0-A_B(4\pi p)^{-1}e^{-p^2}. \tag{49}$$

The ϕ_F term in Eq. (47) represents a smooth modulation of the incident Gaussian beam. As p goes from $-\infty$ to $+\infty$, the phase of ϕ_F changes gradually from π to 0 in a clockwise sense. Therefore ϕ_F is essentially the same as ϕ_1 mentioned above, and can be neglected as far as backscattering effects are concerned. The field ϕ_B is now smooth and bounded, but due to the addition of the term in Eq. (48) it is not strictly a modulated backward propagating leaky wave. The choice of the term added to ψ_B^0 is quite arbitrary as long as it has the proper singular behavior near $p=0$ and possesses no other singularities. We choose the term in Eq. (48) so that the additional field ϕ_F can be discounted.

We now examine ϕ_B in more detail. The asymptotic ($|p|\to\infty$) nature is easily seen by using the alternate form for ψ_B^0, (see Appendix A)

$$\psi_B^0(p)=\begin{cases}-\dfrac{1}{2}A_B\displaystyle\sum_{\substack{-\infty\\(n\neq 0)}}^{\infty}e^{4\pi np}\,\mathrm{erfc}(p+2\pi n), & p>0,\\ A_B-\psi_B^0(-p), & p<0.\end{cases}\tag{50}$$

This, combined with Eqs. (47) and (48) shows that as $x\to-\infty$, ϕ_B and ϕ_2 tend to $A_B\exp(-ik_p w_0 p)$.

Thus A_B is the amplitude of the backward traveling leaky wave. We note that $A_B=O\,[\exp(-q^2)]$. Therefore, the amplitude decreases rapidly as either the beamwidth or the frequency is increased. As $x\to+\infty$, both ϕ_B and ϕ_2 tend to zero like $\exp(-p^2)$. For finite values of x, i.e., near zero, we see from Eqs. (43) and (44) that $\phi_B=O(A_B)$, while $\phi_F=O(1)$. Thus the forward effect by far exceeds the backward propagating effect, as one would expect. In Fig. 2 the contribution from ϕ_B is so small as to be indiscernible. In Fig. 3 we have plotted the magnitude of the quantity $\phi_B(p)/A_B$ for several values of q.

IV. THE BACKSCATTERED FIELD

The total scattered field away from the interface is given by Eq. (23). However, this integral cannot be evaluated in closed form for arbitrary x and z. Instead of using the transform of the total field, $R_0(k_x)\bar{\phi}_{\rm inc}(k_x)$, we will simplify the integral by considering only that part of the interfacial field ϕ_B which produces backward propagating leaky waves. The transform of this field $\bar{\phi}_B$ is composed of all plane-wave contributions, but possesses a maximum near the real wave-

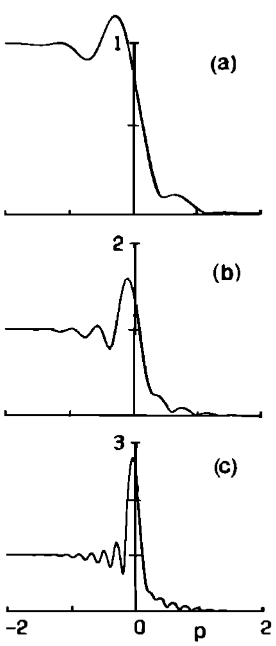

FIG. 3. The magnitude of the backward propagating potential $\phi_B(p)/A_B$ on the interface, as a function of the shifted x coordinate p. (a) $q=4$; (b) $q=8$; (c) $q=16$.

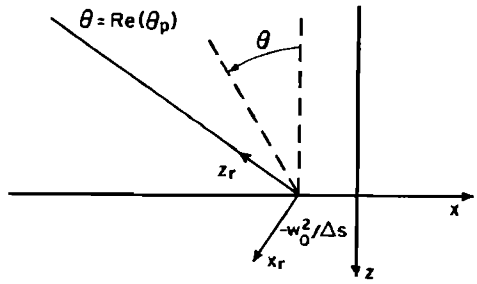

FIG. 4. Coordinate system for the back reflected field.

number $-k_l$ defined in Eq. (34). In Appendix B we have explicitly calculated $\bar{\phi}_B$. In the fluid, $z<0$ we have

$$\phi_B(x,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\phi}_B(k_x)e^{i(k_x x-k_z z)}\,dk_x.\tag{51}$$

The branch of k_z is defined in accordance with the radiation condition, i.e., $k_z=i(k_x^2-k^2)^{1/2}$ if $k_x<-k$ or $k_x>k$.

Let us introduce a polar coordinate system (r,θ), $\theta\epsilon(-\pi/2,\pi/2)$ such that $x+w^2/\Delta s=-r\sin\theta$, $z=-r\cos\theta$. Thus the angle θ is defined in the same sense as the incident angle θ_i but the origin $r=0$ is at $(-w_0^2/\Delta s,0)$, see Fig. 4. Putting $k_x=-k\sin\nu$, we have

$$\phi_B(r,\theta)=\frac{k}{2\pi}\int_C\tilde{\phi}_B(-k\sin\nu)\,e^{ikr\cos(\nu-\theta)}\cos\nu\,d\nu,\tag{52}$$

where the contour C in the complex ν plane goes from $-\pi/2+i\infty$ to $-\pi/2$, then to $\pi/2$ and finally to $\pi/2-i\infty$. Also,

$$\tilde{\phi}_B(-k\sin\nu)\equiv\bar{\phi}_B(-k\sin\nu)e^{i(kw_0^2/\Delta s)\sin\nu}.\tag{53}$$

The poles in the ν plane are given by $\nu=\nu_n$, $n=0,\pm1,\pm2,\ldots$ such that

$$\sin\nu_n=\sin\theta_l+i(\epsilon w_0+4\pi n)/kw_0,\tag{54}$$

where

$$k_l\equiv k\sin\theta_l\tag{55}$$

defines the real leaky wave angle. For large kr, the steepest descents path (SDP) is defined by

$$\nu=\theta+\arcsin[th(u)]-iu,\quad -\infty<u<\infty.\tag{56}$$

In deforming the path of integration to the steepest descents path some poles may be crossed. The poles are at $\nu=\nu_n$, where n are these integers, positive or negative, such that $(\epsilon w_0+4\pi n)/kw_0$ is between 0 and

$$\sec\theta(1-\sin\theta\sin\theta_l)(\sin\theta-\sin\theta_l)\times(1+\sin^2\theta_l-2\sin\theta\sin\theta_l)^{-1/2}.$$

The residue at $\nu=\nu_n$ is

$$\tilde{\phi}_{B,n}\equiv iA_B e^{2\pi n\epsilon w_0}.\tag{57}$$

The net result after deforming the contour is

$$\phi_B(r,\theta)=\frac{k}{2\pi}\int_{\rm SDP}\tilde{\phi}_B(-k\sin\nu)e^{ikr\cos(\nu-\theta)}\cos d\nu+\sum i\tilde{\phi}_{B,n}e^{ikr\cos(\nu_n-\theta)},\tag{58}$$

where the sum is over all eligible n. The pole contributions are easily shown to be evanescent. However the least evanescent term in the sense that $\text{Im}[\cos(\nu_n - \theta)]$ is minimum, corresponds to the pole $n = 0$, which is the backward propagating leaky wave pole. The integral in Eq. (58) may be approximated at large values of kr by the first term in its asymptotic expansion:

$$(k/2\pi r)^{1/2} \cos\theta\, \tilde{\phi}_B(-k\sin\theta) e^{i(kr - \pi/4)}.$$

We note that $\tilde{\phi}_B(-k\sin\theta)$ remains fairly constant for θ between $-\theta_i$ and θ_l but decreases rapidly outside this range. The steepest descents path crosses the leaky wave pole ν_0 when $\theta \sim \theta_l + \epsilon/k$ (assuming $\epsilon \ll k$). Near this angle greater care must be taken in approximating $\phi_B(r,\theta)$. We now discuss a uniform approximation valid in this transition region.

Let us define the complex angle θ_p such that

$$k_p = k\sin\theta_p. \tag{59}$$

Thus $\theta_p = \nu_0$, and for small (ϵ/k) we have $\text{Re}(\theta_p) = \theta_l + O(\epsilon^2/k^2)$. We note that the result that [18]

$$\int_C e^{ikr\cos(\nu-\theta)} \cos\left(\frac{\nu-\theta}{2}\right) \Big/ \left[\sin\left(\frac{\nu-\theta}{2}\right) - \sin\left(\frac{\theta_p-\theta}{2}\right)\right] d\nu$$

$$= 2\pi i e^{ikr\cos(\theta_p-\theta)} \operatorname{erfc}\left[e^{-i\pi/4}(2kr)^{1/2}\sin\left(\frac{\theta_p-\theta}{2}\right)\right]. \tag{60}$$

The uniform contribution to ϕ_B from the pole at $\nu = \theta_p$ may be extracted from Eq. (58) using Eq. (60). The result is $\phi_p(r,\theta)$, where

$$\phi_p(r,\theta) = \tfrac{1}{2}A_B e^{ikr\cos(\theta_p-\theta)} \times \operatorname{erfc}\{e^{-i\pi/4}(2kr)^{1/2}\sin[(\theta_p-\theta)/2]\}. \tag{61}$$

The remainder of $\phi_B(r,\theta)$ follows from Eq. (58) as

$$\phi_B(r,\theta) - \phi_p(r,\theta) = \frac{1}{2\pi}\int_{\text{SDP}}\left\{k\tilde{\phi}_B(-k\sin\nu)\cos\nu + i\phi_p(0,\theta)\cos\left(\frac{\nu-\theta}{2}\right)\left[\sin\left(\frac{\nu-\theta}{2}\right) - \sin\left(\frac{\theta_p-\theta}{2}\right)\right]^{-1}\right\} e^{ikr\cos(\nu-\theta)}\,d\nu + \sum_{n\neq 0} i\tilde{\phi}_{B,n} e^{ikr\cos(\nu_n-\theta)} \tag{62}$$

and the sum is over all eligible n, excluding $n = 0$. The steepest descents integral has no pole at $\nu = \theta_p$. Therefore, for θ near θ_l, the SDP integral will be $O[A_B(kr)^{-1/2}]$.

V. DISCUSSION

The field $\phi_p(r,\theta)$ of Eq. (61) has the same form as that exhibited by a wavefield near a shadow or reflection boundary.[18] In this case we have a boundary defined roughly by $\theta = \theta_l$. For θ less than θ_l, such that $(kr)^{1/2}|\sin[(\theta_p - \theta)/2]|$ is large, we have

$$\phi_p(r,\theta) \sim (8\pi kr)^{-1/2} A_B \csc[(\theta_p - \theta)/2] e^{i(kr + \pi/4)}, \tag{63}$$

which is a diffracted wave that decays with increasing radial distance. However, for θ near or greater than θ_l, the complementary error function in Eq. (61) is of order unity. If $\theta > \theta_l$ and $(kr)^{1/2}|\sin[(\theta_p - \theta)/2]|$ is large, then

$$\phi_p(r,\theta) \sim A_B e^{ikr\cos(\theta_p - \theta)}. \tag{64}$$

This is the backward propagating leaky wave, which we have observed previously on the interface.

We refer to the boundary as a reflection boundary, since the phenomenon is essentially a reflection process, though not specular. The boundary is defined exactly by $\theta = \text{Re}(\theta_p)$ which, as mentioned above, is equal to θ_l correct to order (ϵ^2/k^2). Right on the boundary the amplitude is approximately

$$\phi_p \approx \tfrac{1}{2}A_B e^{ikr} \operatorname{erfc}[e^{i\pi/4}(r\epsilon^2/2k)^{1/2}\sec\theta_l]. \tag{65}$$

The amplitude decreases as the observer goes further into the reflection zone. This is best seen by using Eq. (64) and noting that the imaginary part of $r\cos(\theta_p - \theta)$ is equal to $x_r \text{sh}[\text{Im}(\theta_p)]$, where the reflection coordinates x_r and z_r are defined by Fig. 4. Therefore, the wave is evanescent inside the reflection zone, decaying exponentially with increasing normal distance from the boundary. Hence, for a given r, there must be an angle near θ_l at which the back-reflected field is a maximum.

These observations are made concrete by introducing dimensionless coordinates $\overline{X}$, $\overline{Z}$, and $\overline{R}$ equal to kx_r, kz_r, and kr, respectively. In addition, let $\text{Im}(\theta_p) \equiv \delta$; then we have

$$\left|\frac{\phi_p}{A_B}\right| = \frac{1}{2}e^{-\overline{X}\text{sh}(\delta)}\left|\operatorname{erfc}\left[e^{i\pi/4}(\overline{R}+\overline{Z})^{1/2}\text{sh}\left(\frac{\delta}{2}\right) - \text{sgn}(\overline{X})e^{-i\pi/4}(\overline{R}-\overline{Z})^{1/2}\text{ch}(\delta/2)\right]\right|. \tag{66}$$

For the particular combinations of water–aluminum and water–stainless steel, the value of δ is approximately 0.0173 and 0.0064, respectively.[6] We have considered these two cases for different values of $\overline{Z}$ in Fig. 5. The back reflection beam effect is obvious, whereby the field displays a maximum *inside* the reflection zone ($\overline{X} > 0$). It is also apparent that the effect increases as δ decreases. Now, δ is approximately equal to ϵ/k, and so δ is inversely proportional to the Schoch displacement, see Eq. (35). Therefore, we would expect the effect to increase with increasing Schoch displacement, in agreement with experimental observation.[4] Also, the quantity A_B is essentially proportional to $\exp\{-[w_0(k_i + k_l)/2]^2\}$, since $k\Delta s \gg 1$ (Ref. 6). Thus the effect decreases with increasing beam width w_0, also in agreement with experiment.[4] The above dependence of the backscattered amplitude upon the incident angle [see Eq. (22)] indicates that there is nothing special about the Rayleigh angle of incidence.

VI. CONCLUSION

We have shown that a back-reflected wave exists when a Gaussian beam is incident upon a fluid–solid interface. The reflected wave is due to the backward propagating leaky wave. The reflection boundary is defined by the real part of

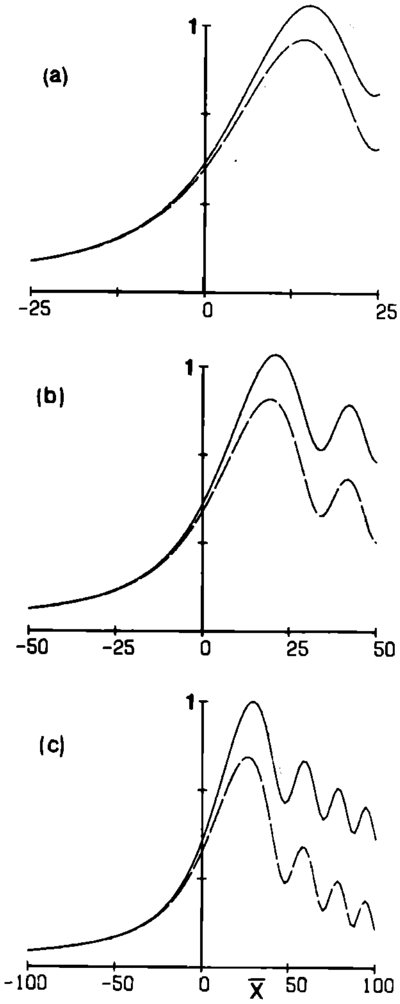

FIG. 5. Magnitude of ϕ_p/A_B for water–stainless steel (solid line) and water–aluminum (dashed line). (a) $\overline{Z}=50$; (b) $\overline{Z}=100$; (c) $\overline{Z}=200$.

the complex angle θ_p and the wave is evanescent inside the reflection zone. In addition, the reflection boundary intersects the interface at a point distant $w_0^2/\Delta s$ from the incident beam center in the backward direction. This type of evanescent zone has not been discussed before and agrees with recent experimental findings. We note that recently described numerical techniques,[19] which have been applied to the forward reflection problem, might also be useful for the back reflection.

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APPENDIX A: AN INFINITE SERIES

Using the Poisson summation formula[20] we have

$$I \equiv \sum_{n=-\infty}^{\infty} \frac{e^{-n^2/4}}{p^2+n^2/4} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-x^2/4-i2\pi n x}}{p^2+x^2/4}\,dx. \tag{A1}$$

By a change of variables, $t=x/2$, and by splitting the denominator into the sum of two simple fractions, we obtain

$$I = \sum_{-\infty}^{\infty} (ip)^{-1} \int_{-\infty}^{\infty} e^{-t^2-i4\pi n t}\left(\frac{1}{t-ip}-\frac{1}{t+ip}\right)dt = \left(\frac{2\pi}{p}\right)e^{p^2}\sum_{-\infty}^{\infty} e^{4\pi n p}\,\mathrm{erfc}(p+2\pi n), \tag{A2}$$

where the integration has been effected using Eq. (26). This result is valid for p real and not equal to zero. Combining this with the definition of ψ_B^0 in Eq. (43) we arrive at Eq. (50).

APPENDIX B: FOURIER TRANSFORM OF ϕ_B

We are interested in finding

$$\overline{\phi}_B(k_x) \equiv \int_{-\infty}^{\infty} e^{-ik_x x}\phi_B(p)\,dx, \tag{B1}$$

where $p = x/w_0 + \epsilon w_0/2$, and ϕ_B is defined by Eq. (47). Let

$$f = w_0(k_x + k_p), \tag{B2}$$

then by Eq. (47),

$$\overline{\phi}_B(k_x) = w_0 e^{i\epsilon k_x w_0^2/2}\int_{-\infty}^{\infty} e^{-ifp}\psi_B(p)\,dp. \tag{B3}$$

From Eqs. (48) and (50) we have,

$$\int_{-\infty}^{\infty} e^{-ifp}\psi_B(p)dp = \frac{iA_B}{f} - 2i\int_0^{\infty}\psi_B^0(p)\sin(fp)dp - \left(\frac{iA_B}{2\pi}\right)\int_0^{\infty} e^{-p^2}\sin[(f-f_i)p]\,\frac{dp}{p}, \tag{B4}$$

where

$$f_i = w_0(k_i + k_p). \tag{B5}$$

The final integral in Eq. (B4) is (Ref. 21, Eq. 3.896.4)

$$\int_0^{\infty} e^{-p^2}\sin[(f-f_i)p]\,\frac{dp}{p} = \left(\frac{\pi}{2}\right)\mathrm{erf}\left(\frac{f-f_i}{2}\right). \tag{B6}$$

The second integral in Eq. (B4) can be done by substituting from Eq. (50) for ψ_B^0 and using the result

$$\int_0^{\infty}\sin(fp)[e^{2gp}\,\mathrm{erfc}(p+g)+e^{-2gp}\,\mathrm{erfc}(p-g)]dp = [2f/(f^2+4g^2)](1-e^{-(g^2+f^2/4)}), \tag{B7}$$

which can be verified by partial integration. Combining Eqs. (B3)–(B7), we obtain

$$\overline{\phi}_B(k_x) = iw_0A_Be^{i\epsilon k_x w_0^2/2}\Bigg(\frac{1}{f}+\sum_{n=1}^{\infty}\frac{2f}{f^2+(4\pi n)^2}\times\{1-\exp-[(2\pi n)^2+f^2/4]\} - \tfrac{1}{4}\mathrm{erf}[(f-f_i)/2]\Bigg). \tag{B8}$$

This can be further simplified by noting that

$$\sum_{-\infty}^{\infty}\frac{x}{x^2+n^2} = \pi\coth(\pi x) \tag{B9}$$

and observing that the exponential sum in Eq. (B8) is negligibly small. Thus

$$\overline{\phi}_B(k_x) = \tfrac{1}{4} i\, w_0 A_B e^{i\epsilon k_x w_0^2/2} \{\coth(f/4) - \mathrm{erf}[(f - f_i)/2]\} \tag{B10}$$

with a relative error of $\sim e^{-40}$.

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