Inverse ray tracing in elastic solids with unknown anisotropy

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The problem of inverse ray tracing in a homogeneous anisotropic elastic solid is considered. The wave speeds in the solid are assumed unknown, and must be obtained in the course of the inversion. The specific problem of locating a crack tip in a two-dimensional geometry is investigated. The data are assumed to be in the form of travel times of diffracted ultrasonic signals between transducers positioned on an exterior surface of the solid. Both pulse–echo and pitch–catch data are considered. It is found that travel-time data on the exterior surface suffices to locate the crack tip only if the material is isotropic. If the material is anisotropic, we must be able to move the source and/or receiver in the direction normal to the surface. The same problem is considered with the source and receiver positioned in a surrounding isotropic material, e.g., a water bath. It is shown that the ray inversion is now possible only if the solid is isotropic, the problem being underdetermined for an anisotropic solid. This indicates that the problem of inverse ray tracing, in the context of crack sizing, is not possible in a medium which is both inhomogeneous and anisotropic. Numerical results are presented for a synthetic experiment in which a finite crack is present in some transversely isotropic homogeneous elastic solids. It is demonstrated that an initial presumption of isotropy can lead to very erroneous results.

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INTRODUCTION

The problem of inferring the size and location of cracks in metals is of great importance in nondestructive evaluation. Both low- and high-frequency ultrasonic methods have been considered for the corresponding inverse scattering problem. In this paper we assume that the scattering is in the high-frequency regime, i.e., that the scattered signals propagate along rays.[1]

The ray paths in a homogeneous anisotropic elastic solid are straight lines. However, certain distinctions must be made between the actual signal velocity and the corresponding phase velocity. (Note: by velocity we mean a vector quantity, its magnitude is the speed.) The signal velocity, also known as the group velocity or energy propagation velocity, is the more important quantity, although for linear elasticity the phase velocity is easier to determine.[2] Generally the two velocities are not equal. It may be shown that they are equal if they are parallel, or if the speeds are equal. We note that the distinctions of velocities is redundant in isotropic solids. Additional complications occur when the signal velocity is locally concave in a given direction, giving rise to the phenomenon of conical refraction.[3] However, even in this case it can be shown[4] that discontinuities propagate in straight lines with the corresponding signal velocity.

The direct scattering of elastic waves by cracks has been studied extensively. In the high-frequency regime, i.e., when the incident wavelength is small compared with the characteristic dimensions of the crack, it can be shown that the major contribution to the scattered field comes from certain "flash points" on the crack edge. These flash points may be predicted by the geometrical theory of diffraction as applied to elastic waves. For more details we refer the reader to Ref. 5. Ultrasonic experiments on the scattering of elastic waves by cracks have confirmed the existence of such flash points.

The purpose of this paper is to find the necessary and sufficient conditions for finding a flash point in an anisotropic elastic medium whose anisotropy is not known *a priori*. The experimental procedure envisaged is as follows: a source transducer S emits a signal, which is diffracted at the flash point F and subsequently is observed by the receiver transducer Q. The scattering may be pulse–echo, in which case the source and receiver are identical, or pitch–catch, when separate source and receiver are used. It is assumed in either case that the travel time of the diffracted signal can be measured.

We begin with the simplest configuration, where the scattering is pulse–echo and the source–receiver S is positioned on an exterior surface of the material which is assumed homogeneous. We consider the different situations where the material is isotropic or anisotropic, and the wave speeds are known or not known *a priori*. Next, we treat the same problem when the scattering is pitch–catch. The situation is now more complicated than for pulse–echo, since there are two ray paths present instead of one. We also consider the case when pitch–catch and pulse–echo data are available simultaneously. Some examples of inverse ray tracing using synthetic data are presented in Sec. IV.

It is common practice in ultrasonic experiments to position the transducers in a water bath surrounding the specimen. The ray tracing must now account for the fluid–solid interface. In Sec. III we consider the fluid–solid configuration and note some interesting consequences for the inverse ray tracing problem in anisotropic inhomogeneous media. First we state a basic result which will be utilized in later sections.

I. A RESULT BASED ON FERMAT'S PRINCIPLE

In a dispersionless medium, high-frequency signals propagate along rays which satisfy Fermat's Principle. Thus the diffracted rays describe curves in space which make the travel time from source to receiver stationary with respect to all neighboring curves. This is just the well-known correspondence between high-frequency signals and wave fronts.

In a homogeneous anisotropic elastic medium the rays are straight lines and the number of wave speeds in a given direction $\mathbf{p}$, $|\mathbf{p}| = 1$, is equal to the number of sheets of the wave surface in that direction, e.g., two if the material is isotropic. Now suppose that a scatterer (e.g., a crack) is present in the material and consider the diffracted ray from the source S via the flash point F to the observer Q. The associated travel time of the ray is

$$T = |SF|/c_1 + |FQ|/c_2, \tag{1}$$

where c_1 and c_2 are the wave speeds of the rays from S to F and from F to Q, respectively. The two speeds may correspond to the same sheet of the wave surface or they may be from different sheets, in which case mode conversion occurs at diffraction.

Let M signify either one of the points S or Q, the source and receiver points. Define the unit vector $\mathbf{p}$ as direction vector from M to the flash point F. The signal speed c of the ray between M and F is dependent on the direction $\mathbf{p}$, thus $c = c(\mathbf{p})$. Our basic result is that the spatial gradient ∇T of the travel time of the diffracted signal with respect to the position M is related to the direction $\mathbf{p}$ by

$$\nabla T = (-1/c)[\mathbf{p} - (1/c)\overline{\nabla}c], \tag{2}$$

where $\overline{\nabla}$ is the angular gradient operator for the unit vector $\mathbf{p}$. For example, if $c(\mathbf{p}) = c(\theta,\phi)$ where θ,ϕ are spherical polar angles, then

$$\overline{\nabla}c = \mathbf{e}_\theta \frac{\partial c}{\partial \theta} + \csc\theta\, \mathbf{e}_\phi \frac{\partial c}{\partial \phi}. \tag{3}$$

The proof of Eq. (2) can be found in the Appendix of Ref. 6. We note that if the material is isotropic, $\overline{\nabla}c \equiv 0$, and Eq. (2) has a particularly simple and obvious form. Also, the above result can be shown to hold for inhomogeneous media, where the rays are not straight.

If we let the right-hand side of Eq. (2) be $-s\mathbf{n}, |\mathbf{n}| = 1$, then $s\mathbf{n}$ is the slowness vector corresponding to the wave velocity $c\mathbf{p}$. The corresponding phase velocity is $(1/s)\mathbf{n}$. In geometrical terms, the vectors $c\mathbf{p}$ and $s\mathbf{n}$ are polar reciprocal to each other. Also, since $\mathbf{p}\cdot\overline{\nabla} \equiv 0$ by definition, we have

$$|\nabla T| = s = (1/c)[1 + (1/c^2)|\overline{\nabla}c|^2]^{1/2} \geq 1/c, \tag{4}$$

with equality only if the material is (locally) isotropic.

II. RAY TRACING IN A SOLID ONLY

As discussed before, there are two arrangements possible in order to generate a single piece of scattering data: the pulse–echo or the pitch–catch arrangement. We first consider pulse–echo data. To further simplify the problem, the geometry is taken as two dimensional. This could correspond to a transversely isotropic material in which all rays propagate in a plane containing the axis of symmetry.

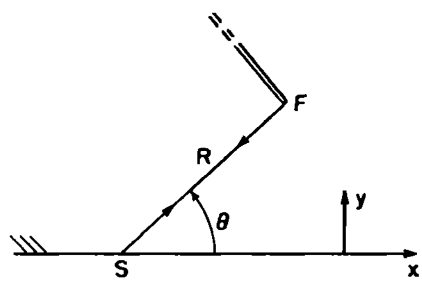

FIG. 1. Two-dimensional pulse–echo arrangement. The flash point is the crack tip F.

A. Pulse–echo data

Consider a transducer S positioned on the exterior surface of a specimen which contains an interior crack tip F, see Fig. 1. A signal is emitted from S, diffracted at F, and subsequently received at S after a time delay of $2T$. Let us assume that the speeds of propagation to and from the flash point are identical. Thus

$$T = R/c, \tag{5}$$

where $c = c(\theta)$. Application of Eq. (2) gives

$$T_x = -(1/c)(\cos\theta + \gamma\sin\theta), \tag{6}$$

$$T_y = -(1/c)(\sin\theta - \gamma\cos\theta), \tag{7}$$

where

$$\gamma(\theta) \equiv \frac{1}{c}\frac{dc(\theta)}{d\theta},$$

and T_x, T_y are the derivatives of T at S. Further differentiation of Eqs. (6) and (7) gives

$$T_{xx} = (1/Rc)\sin^2\theta(1 + \gamma^2 - \gamma'), \tag{8}$$

$$T_{yy} = (1/Rc)\cos^2\theta(1 + \gamma^2 - \gamma'), \tag{9}$$

$$T_{xy} = -(1/Rc)\cos\theta\sin\theta(1 + \gamma^2 - \gamma'), \tag{10}$$

where $\gamma' = d\gamma/d\theta$. The derivatives T_x and T_{xx} can be computed by shifting the transducer tangentially on the surface at S, to some new point S_1. The travel time of the pulse–echo signal is measured at S_1, say T_1. Then a finite difference approximation to T_x can be formed. Similarly, a finite difference procedure can be used to approximate T_{xx} if two shifts are performed. The derivatives in the y direction are more difficult to handle experimentally. An approximate evaluation of T_y, T_{xy}, or T_{yy} by any finite difference scheme requires shifting the transducer in the direction normal to the solid surface. One way of achieving this is by introducing a thin slab of the same material at S such that the principal directions of the slab and the specimen are exactly aligned.

We now consider the necessary and sufficient data required to find the flash point F. Suppose first that the medium is either isotropic or anisotropic and that the speed $c(\theta)$ is known as a function of angle. Then a knowledge of T and T_x provides enough information to determine the two unknowns R and θ by Eqs. (5) and (6).

Next, suppose that the medium is isotropic but the speed c (a constant) is unknown. Now there are three unknowns R, θ, and c, and we require three pieces of information. In Eqs. (6)–(10) we have $\gamma = \gamma' = 0$. Therefore we may use Eqs. (5) and (6) and either of Eqs. (7) and (8). However, use of Eq. (7) means computing the y derivative of T, which is much more cumbersome experimentally. Using T, T_x, and T_{xx} we have $R = cT$ and

$$\theta = \tan^{-1}[-(TT_{xx})^{1/2}/T_x], \tag{11}$$

$$c = [2/(T^2)_{xx}]^{1/2}. \tag{12}$$

If the medium is anisotropic with unknown speed $c(\theta)$, there is no way to avoid using y derivatives of T. By inspection of Eqs. (5)–(10), it is apparent that since γ and γ' are nonzero, we must solve for the five unknowns R, c, θ, γ, and γ'. Thus we require knowledge of five of the six quantities on the left-hand sides of Eqs. (5)–(10). We choose as our quantities T, T_x, T_y, T_{xx}, and T_{xy}, since these do not involve second derivatives with respect to y. We find that

$$c = (T_{xx}^2 + T_{xy}^2)^{1/2}/|T_{xx}T_y - T_{xy}T_x|, \tag{13}$$

$$\theta = \tan^{-1}(-T_{xx}/T_{xy}), \tag{14}$$

and the other quantities follow simply.

1. Small anisotropy

Let the medium be almost isotropic in the sense that both γ and γ' are small quantities. Again, consider the situation where the speed is not known *a priori*. If we consider the three quantities T, T_x, and T_{xx}, and invert Eqs. (5), (6), and (8) neglecting γ and γ', the results for R, θ and c are as in Eqs. (11) and (12), which we call R_0, θ_0, and c_0. A more detailed inversion of Eqs. (5), (6), and (8) reveals

$$R = R_0[1 + \epsilon \sin\theta_0 \cos(\theta_0 + \phi) + O(\epsilon^2)], \tag{15}$$

$$\theta = \theta_0 + \epsilon \sin\theta_0 \sin(\theta_0 + \phi) + O(\epsilon^2), \tag{16}$$

$$c = c_0[1 + \epsilon \sin\theta_0 \cos(\theta_0 + \phi) + O(\epsilon^2)], \tag{17}$$

where $\phi = \tan^{-1}(\gamma'/2\gamma)$ and $\epsilon \equiv [\gamma^2 + (\gamma'/2)^2]^{1/2} \ll 1$. Thus, if the data is inverted using the method for isotropic materials, the errors in the computed quantities R, θ, and c will be given by Eqs. (15)–(17) for small anisotropy.

Now consider the three quantities T, T_x, and T_y, and invert Eqs. (5)–(7) for small $|\gamma|$. We find

$$\theta = \tan^{-1}(T_y/T_x) + \gamma + O(\gamma^3), \tag{18}$$

$$c = (T_x^2 + T_y^2)^{-1/2}[1 + O(\gamma^2)], \tag{19}$$

and $R = cT$. Therefore, if we are willing to use one y derivative, the error in R and c can be made to be $O(\epsilon^2)$ for small anisotropy. However, whether we use T_y or T_{xx}, the error in the computed flash point position is of order (ϵR).

Finally, before we leave the discussion of the pulse–echo problem, we remark on the problem when mode conversion occurs. In this case the signal has different speeds c_1 and c_2 as it propagates to and from the flash point. Equations (5)–(10) are replaced by similar equations with the term $(1/2)(1/c_1 + 1/c_2)$ substituted for $1/c$. Therefore any inversion of these equations cannot produce c_1 and c_2 separately, but only in the combination $(1/c_1 + 1/c_2)$.

B. Pitch–catch data

A source S and a receiver Q are positioned a distance l apart on the surface of the specimen, which we have taken as flat for convenience, see Fig. 2. The total travel time for the

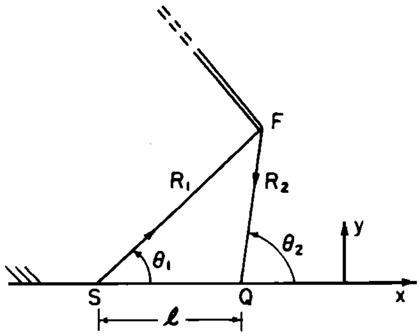

FIG. 2. Two-dimensional pitch–catch configuration.

diffracted signal from S to Q is

$$T = R_1/c_1 + R_2/c_2, \tag{20}$$

where $c_1 = c_1(\theta_1)$ and $c_2 = c_2(\theta_2)$ are the two wave speeds on the rays. Analogously to Eqs. (6)–(10) we have

$$T_{jx} = -(1/c_j)(\cos\theta_j + \gamma_j \sin\theta_j), \quad j = 1,2, \tag{21a,b}$$

$$T_{jy} = -(1/c_j)(\sin\theta_j - \gamma_j \cos\theta_j), \quad j = 1,2, \tag{22a,b}$$

$$T_{1xx} = (1/R_1c_1)\sin^2\theta_1(1 + \gamma_1^2 - \gamma_1'), \tag{23}$$

$$T_{1xy} = -(1/R_1c_1)\sin\theta_1 \cos\theta_1(1 + \gamma_1^2 - \gamma_1'), \tag{24}$$

where the subscripts 1 and 2 indicate that the derivatives are evaluated at S and Q, respectively. In addition, we have the two geometrical identities:

$$0 = R_1 \sin\theta_1 - R_2 \sin\theta_2, \tag{25}$$

$$l = R_1 \cos\theta_1 - R_2 \cos\theta_2, \tag{26}$$

l being the known distance between source and receiver.

If the constitutive nature of the specimen is known *a priori*, i.e., the speeds are known functions of angle, then the flash point may be found from T and (say) T_{1x}. Things become more complicated if the ray speeds are unknown.

First, suppose that the material is isotropic and the speed on both rays is the same, say c. Then three pieces of data are required. The three simplest to obtain are T, T_{1x}, and T_{2x}, involving no normal or second derivatives. We now have five equations (20), (21a), (21b), (25), and (26) for five unknowns, $R_1, R_2, \theta_1, \theta_2, c$. Equations (21a) and (21b) give

$$c = -\cos\theta_1/T_{1x}, \tag{27}$$

$$\cos\theta_2 = (T_{2x}/T_{1x})\cos\theta_1. \tag{28}$$

While Eqs. (20), (25), and (26) give

$$R_1 = l \sin\theta_2 \csc(\theta_2 - \theta_1) \tag{29}$$

$$R_2 = cT - R_1, \tag{30}$$

$$\tan\theta_1 = (1 + \alpha)|\alpha^2 + 2\alpha T_{1x}/(T_{1x} - T_{2x})|^{-1/2}, \tag{31}$$

where $\alpha = lT_{1x}/T$.

Next, suppose the material is again isotropic with speeds unknown, but the diffracted ray is mode converted. For example, the incident ray might be one of longitudinal motion, and the diffracted ray of transverse motion. We need one additional piece of information compared with the previous case, since there are now six unknowns: R_j, θ_j, and $c_j, j = 1,2$. In order to avoid using normal derivatives, we choose T_{1xx} as the extra datum. Thus the left-hand sides of Eqs. (20), (21a), (21b), (23), (25), and (26) are all known. The unknowns are found as follows: Eqs. (20), (21a), (21b), and (25) give R_1, R_2, c_1, and c_2 in terms of the angles θ_1 and θ_2, which are determined by the remaining two equations. We find that

$$\tan \theta_2 = \tan \theta_1/(1 + \beta \csc^2 \theta_1), \tag{32}$$

$$\sin^2 \theta_1 = \beta [\delta - \rho + (\rho^2 - \beta^2 \delta)^{1/2}]/(2\rho - \delta - \beta^2), \tag{33}$$

where

$$\beta = lT_{1xx}/T_{1x}, \quad \delta = (TT_{1xx}/T_{1x}T_{2x}) - \beta,$$

$$2\rho = \delta(1 + \beta) + \beta(1 + T_{1x}/T_{2x}).$$

The problem of finding the flash point becomes more complicated if the material is anisotropic with unknown speed dependence upon angle. The terms γ_1, γ_2, and γ_1' in Eqs. (21) and (23) are nonzero, and are essentially unknowns. Thus there are nine unknowns, $R_j, \theta_j, c_j, \gamma_j, j = 1,2$, and γ_1', requiring all nine equations (20)–(26) to solve them. The minimum required data for inversion is $T, T_{jx}, T_{jy}, j = 1,2, T_{1xx}$, and T_{1xy}. We note that we could use T_{1yy} instead of T_{1xy}, but the latter requires only a first derivative normal to the surface. In either case, it is obvious that normal derivatives are necessary to obtain a closed system of equations. The unknowns are obtained as follows: from Eqs. (23) and (24) we may solve for θ_1 and $(\gamma_1^2 - \gamma_1')$ explicitly. Then Eqs. (21a) and (22a) give us c_1 and γ_1. The remaining equations readily give R_1, R_2, γ_2, and c_2 in terms of θ_2, where θ_2 satisfies

$$\tan \theta_2 = \tan \theta_1 (T - lT_{2x})/(T + lT_{2y} \tan \theta_1 - l \sec \theta_1/c_1). \tag{34}$$

All of the above problems requiring pitch–catch data can be simplified a great deal if we also have simultaneous access to pulse–echo data.

C. Pulse–echo and pitch–catch data

The same geometry as for pitch–catch problem is considered, see Fig. 2. The only difference is that now the source also acts as a receiver, able to measure the time delay of the diffracted ray from S to F and back to S again. The pulse–echo diffracted signal is assumed not to be mode converted, so that it propagates with the same speed on the two rays SF and FS. We now have two equations,

$$T_j = R_j/c_j, \quad j = 1,2, \tag{35a,b}$$

instead of the single Eq. (20), where T_1 and T_2 are the travel times on the rays SF and FQ, respectively. Measurement of the pitch–catch travel time gives $T_1 + T_2$, while the pulse echo time gives $2T_1$. Thus we assume that both T_1 and T_2 are known.

The flash point inversion is now particularly simple if the material is isotropic, with known speeds c_1 and c_2. The flash point is the intersection of two circles centered at S and Q having radii of c_1T_1 and c_2T_2, respectively. If the material is isotropic with unknown speeds, but the signal is not mode converted, then the single unknown speed c can be found from one additional piece of information. Using the tangential derivative T_{1x}, for example, gives

$$c = l(T_2^2 - T_1^2 - 2lT_1T_{1x})^{-1/2}. \tag{36}$$

Similarly, if the signal is mode converted, the two unknown speeds c_1 and c_2 can be obtained if we know T_{2x} as well as T_{1x}. We find that

$$c_1 = (l/T_1)|(1 - \alpha_2)/(\alpha_2 - \alpha_1 + 2\alpha_1\alpha_2)|^{1/2}, \tag{37}$$

where $\alpha_j = lT_{jx}/T_j, j = 1,2$, and c_2 is got by interchanging 1 and 2.

So far we have not required taking normal derivatives. If the material is anisotropic with unknown speeds, the quantities γ_1 and γ_2 in Eqs. (21) are nonzero, and thus additional unknowns. The simplest closed system of equations is obtained by using the two equations for T_{1y} and T_{2y}. Now we have eight unknowns, R_j, θ_j, c_j, and $\gamma_j, j = 1,2$ and eight equations (21a), (21b), (22a), (22b), (25), (26), (35a), and (35b). Note that we have not required second derivatives of T, making Eqs. (23) and (24) redundant. Solving the eight equations gives the speeds as

$$c_j = -1/(T_{jx} \cos \theta_j + T_{jy} \sin \theta_j), \quad j = 1,2, \tag{38}$$

where the angle θ_j is given by

$$\tan \theta_j = (\alpha_1 - \alpha_2 - \alpha_1\alpha_2)/(\beta_2 - \beta_1 + \beta_j\alpha_{3-j}), \quad j = 1,2. \tag{39}$$

Here $\alpha_j = lT_{jx}/T_j$, and $\beta_j = lT_{jy}/T_j, j = 1,2$. Having found the speeds, the flash point is found as the intersection of the two circles centered at S and Q of radii c_1T_1 and c_2T_2, respectively.

III. RAY TRACING THROUGH AN INTERFACE

Ultrasonic experiments are often performed in a water bath. The ultrasonic signal originates from and is received by transducers in the water, which are not in direct contact with the specimen. The signals scattered by flaws in the specimen must cross the fluid–solid interface twice before they can be detected by the receiver transducer. Therefore we consider the fluid–solid configuration as a model for the problem of ray tracing when an interface is present.

For simplicity, the scattering is assumed to be pulse–echo and the interface is taken as flat. The geometry is again two dimensional. The source S is a distance b from the interface, which is the exterior surface of the material, see Fig. 3. The pulse–echo diffracted ray makes angles θ_f and θ with the interface in the fluid and in the solid, respectively. Let $2T$ be the time taken for the signal to propagate from S to F and back again. Referring to Fig. 3, we have the relationships

$$b = R_f \sin \theta_f, \tag{40}$$

$$T = R_f/c_f + R/c, \tag{41}$$

where subscript f indicates a quantity in the fluid. The fluid is assumed to be homogeneous and isotropic, so that c_f is constant. The wave speed in the solid, may, because of anisotropy, depend upon θ. Snell's law at the interface is

$$\cos \theta_f/c_f = \cos \theta (1 + \gamma \tan \theta)/c, \tag{42}$$

where $\gamma(\theta)$ is as before. Equation (42) can be derived from Fermat's Principle as follows: the stationary nature of the

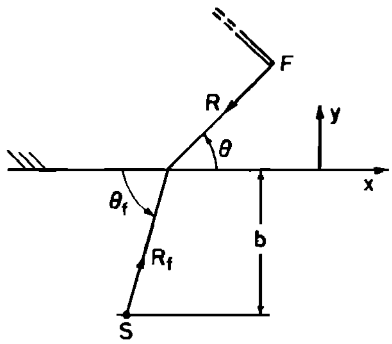

FIG. 3. Pulse–echo arrangement with the source–receiver in water.

ray paths can be shown to imply that the tangential derivative of travel time T be continuous across the interface. This, in conjunction with Eq. (2) gives Eq. (42). The spatial gradient of T at the source–receiver S is by Eq. (2),

$$\nabla T = -\mathbf{p}/c_f, \tag{43}$$

since the fluid is isotropic. Here $\mathbf{p} = (\cos\theta_f, \sin\theta_f)$. Assuming c_f known, the angle θ_f is determined from Eq. (43) by any single component of ∇T. Having found θ_f, the distance R_f is got from Eq. (40). Thus a knowledge of T and a single first-order derivative of T (two if c_F is not known *a priori*) determines the quantities R/c and $(\cos\theta + \gamma\sin\theta)/c$ by Eqs. (41) and (42). This is analogous to ray tracing in a solid only, in which we know the left-hand sides of Eqs. (5) and (6).

Now consider second-order derivatives of T with respect to the position S. If $\mathbf{n}$ is any direction vector, we have that

$$(\mathbf{n}\cdot\nabla)^2 T = |\mathbf{n}\wedge\mathbf{p}|^2/c_f a, \tag{44}$$

where a is the radius of curvature of the wave front at S. It is straightforward to show that

$$a = R_f + R\left(\frac{\sin\theta_f\tan\theta_f}{\sin\theta\tan\theta}\right)\left(\frac{1+\gamma\tan\theta}{1+\gamma^2-\gamma'}\right), \tag{45}$$

where $\gamma'(\theta)$ is as before. Assume that a is known by measurement of some second-order derivative of T. Also, the quantity $z \equiv (\cos\theta + \gamma\sin\theta)/c$ is known from first derivatives. We note from Eq. (45) that the quantity $z\sin\theta_f\tan\theta_f/(a - R_f)$ is precisely the right-hand side of Eq. (8). Thus we have reduced the ray inversion problem in the fluid–solid to a similar problem in the solid only, where the quantities T, T_x, and T_{xx} of Eqs. (5), (6), and (8) are known. But, as discussed above, this information is sufficient for finding the flash point only if the solid is isotropic, see Eqs. (11) and (12). If the material is anisotropic, further information is required, specifically the left-hand sides of Eqs. (7) and (9) or (10). These quantities all involve normal derivatives of the travel time at the interface. However, according to Eq. (42), (Snell's law or Fermat's Principle) on traversing the interface, the ray transmits only information concerning the tangential derivative. All information about the normal derivatives in one medium is lost as the ray enters a different medium. We conclude that the general problem of ray inversion in unknown anisotropic media is underdetermined if the measurements are made only within some other medium.

This result has important ramifications for the problem of inverse ray tracing in an inhomogeneous anisotropic solid. For we may generalize the result to a stratification of piecewise homogeneous anisotropic layers. By letting the number of layers become infinite, we obtain a stratified anisotropic inhomogeneous body. Hence, ray inversion in its present context is not possible in an anisotropic inhomogeneous body. We note that inverse ray tracing is possible in an inhomogeneous stratified half-space which is isotropic, as in the Wiechert–Herglotz problem of geophysics.[7] Therefore inverse ray tracing is possible in the presence of either anisotropy or inhomogeneity, but not in the presence of both simultaneously.

IV. A NUMERICAL EXAMPLE USING PITCH–CATCH AND PULSE–ECHO DATA

We now consider the inversion of synthetic data for the two-dimensional configuration in which both pulse–echo and pitch–catch signals are measured on the surface of an anisotropic solid. The constitutive nature of the anisotropy is taken to be that of several different transversely isotropic elastic materials. Both exact and approximate inversion schemes are considered, and their accuracy is investigated in terms of the finite difference scheme employed.

Let a transversely isotropic elastic solid occupy the half-space $y \geqslant 0$, with its symmetry axis in the y direction. Define the dimensionless numbers ϵ_1 and ϵ_2 by

$$\epsilon_1 = (C_{33} - C_{11})/2C_{11}, \tag{46a}$$

$$\epsilon_2 = (C_{13} + 2C_{44} - C_{11})/C_{11}, \tag{46b}$$

where C_{11}, C_{33}, C_{44}, and C_{13} are the usual elastic constants, see for example, Ref. 2. If $\epsilon \equiv \max(|\epsilon_1|, |\epsilon_2|)$ is small, then it may be shown that the *fastest* wave speed $c(\theta)$ in the direction θ is given by

$$c(\theta)/c(0) = 1 + (\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta)\sin^2\theta + O(\epsilon^2), \tag{47}$$

where θ is the angle between the surface $y = 0$ and the propagation direction.[8] We note that in an isotropic solid, $\epsilon_1 = \epsilon_2 = 0$ and $c(\theta) = c_L$, the longitudinal wave speed. Values of ϵ_1 and ϵ_2 for some solids with hexagonal crystal structure are given in Table I. These were obtained from the values for the corresponding elastic constants given in Ref. 2. In the following examples we have used the expression in Eq. (47) with terms of order ϵ^2 neglected to calculate the wave speed as a function of angle. The value of the wave speed has been nondimensionalized by taking the speed tangential to the surface, i.e., $c(0)$, equal to unity.

A flat crack is assumed to lie in the (x,y) plane with one tip located at the point (0,1) and the other tip at (0,1) $+ d(\cos\psi, \sin\psi)$, where $d = 0.5$ and $\psi = 45°$. We have cho-

TABLE I. Values of the dimensionless numbers ϵ_1 and ϵ_2 for several materials with hexagonal crystal structure (from Musgrave[2]).

Material	ϵ_1	ϵ_2
Beryl	−0.08	−0.25
Beryllium	0.08	0.16
Magnesium	0.02	−0.25
Zinc	−0.30	−0.19
Cadmium	−0.28	−0.30

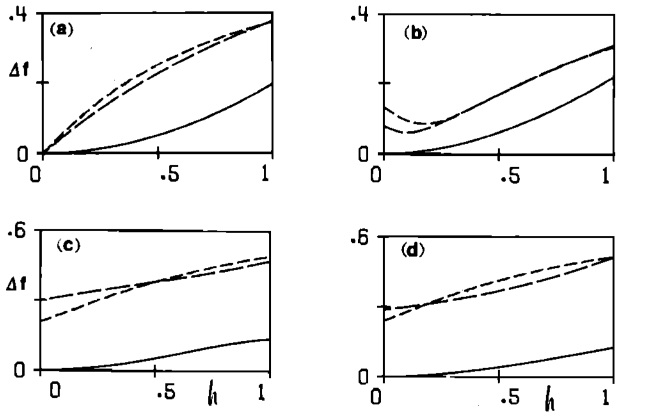

FIG. 4. Error (Δf) in the estimated flash point position versus the shift distance h for various anisotropic solids: (a) isotropic, (b) beryllium, (c) zinc, and (d) cadmium. Method (A) ————————; method (B)— — —; method (C) - - - - - - - -.

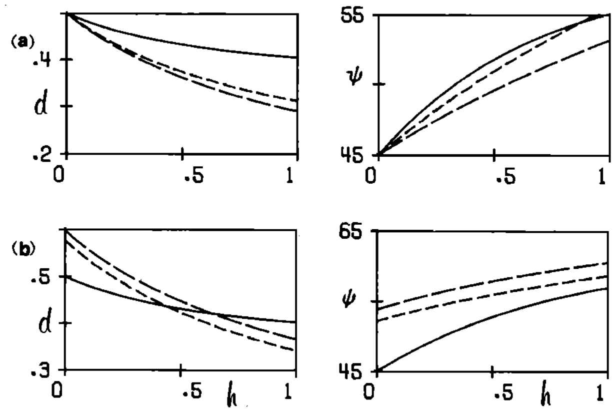

FIG. 5. Estimated crack length d and orientation ψ versus shift distance h (correct values 0.5 and 45°), (a) isotropic, (b) beryllium. Method (A) ————; method (B) —— —— —— ——; method (C) - - - - - - - - - -.

sen the source and observer locations on the surface $y = 0$ to be at $x = 0$ and $x = 1$, respectively. Thus the first diffracted signals are from the flash point at (0,1).

A "synthetic" experiment is performed as follows: for a given material, the pulse–echo and pitch–catch travel times T_1 and T_2 are computed using Eq. (47) with $c(0) = 1$. The observation point is shifted a distance h in the x direction, and the new pitch–catch time T'_2 is computed. The quantity $(T'_2 - T_2)/h$ is then used as the "experimental" value of $\partial T_2/\partial x$. Similarly $\partial T_1/\partial x$, $\partial T_1/\partial y$, and $\partial T_2/\partial y$ are approximated by finite differences, with the shift being equal to h in each case. The position of the flash point at (0,1) is estimated by three different methods and the dependence of the error on the shift distance h is checked. The three methods employed are those described in Sec. IIC:

(A) The exact solution, Eqs. (38) and (39). For this we require all four partial derivatives $\partial T_j/\partial x, \partial T_j/\partial y, j = 1,2$, and we would expect exact agreement as $h \rightarrow 0$.

(B) The isotropic solution, with two different wave speeds, Eq. (37). This uses the two x derivatives only.

(C) The isotropic solution with only one wave speed, Eq. (36), for which only one x derivative is needed.

In Fig. 4 we have plotted the error, defined as the distance between the estimated tip and the actual tip (0,1), as a function of shift distance h for four different anisotropies. Beryllium was chosen as an example of slight anisotropy, while zinc and cadmium are highly anisotropic. The isotropic case is shown for comparison. We note that the exact method exhibits a linear error growth in all examples. It is to be noted that method C, which uses the least amount of information, gives almost the same results as method B.

In Fig. 5 we have plotted the estimated crack length and orientation, the exact values being 0.5° and 45°, respectively. The curves in Fig. 5 were calculated by estimating the second flash point in a similar manner to that used for the near flash point. The results shown for beryllium were typical of the materials considered. Thus an initial assumption of isotropy can lead to relatively large errors in the crack size and orientation, even when we have exact travel times and derivatives ($h \rightarrow 0$) available.

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