The simplest complex periodic hamiltonian only has $\hbar/2$ solutions

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An exact wave function is derived from a self-consistent quantum Hamiltonian in 2+1 dimensions. It is shown that in order to produce a self-consistent solution for a central, n-fold azimuthally symmetric, potential in the plane, there is no freedom for the radial form of the potential, for the wave number of the solution, or for the angular momentum of a bound state. The solution has three free parameters, is mathematically required to live in a $1/r$ potential, to exhibit exactly three varieties, and to have angular momentum prescribed identically to be $\hbar/2$. No other self-consistent solutions are possible.

PACS Nos.: 04.20.Jb, 11.10.Ef, 12.40.-y
It has been known for nearly two decades that the contribution to the angular momentum of the proton is not accounted for by the intrinsic spins of constituent quarks, and must presumably be associated with orbital angular momentum [1]. This ‘spin crisis’ [2] has prompted numerous theoretical, computational, and experimental investigations [3]. In this letter, we present the first analytic formulation of the problem in terms of the simplest possible quantum model of n-interacting particles in a central, but azimuthally varying, field. As we will show, this formulation has three possible exact solutions, and surprisingly these solutions can only exist if the orbital angular momentum of the system is an integer multiple of \( \hbar/2 \). This suggests that the proton’s spin may be entirely due to the orbital angular momentum of its components. We derive this result, and then discuss implications.

We begin by displaying a classical cartoon of the problem under consideration in Fig. 1. Here we consider a number, \( n \), of discrete particles orbiting a common center, \( C \), each with velocity \( V \). One of the simplest Hamiltonians that could describe this system would be \( n \)-fold azimuthally periodic and two-dimensional, i.e.:

\[
H_I(r, \theta, t) = P(r) \exp[i(k\theta - \omega t)] ,
\]

where \( k \) and \( \omega \) are real constants, \( k = 2\pi/n \), and \( P(r) \) is a radial interaction, which is to be determined. Thus in this cartoon, we depart from the usual central potential formulation, in which a prescribed central potential confines particles (e.g. in the Bohr model); instead, we prescribe that the central potential should vary with \( \theta \) so that particles orbiting the center, \( C \), with angular speed \( \omega/k \) always experience an unchanging potential. We do not describe at this point how such a potential would be maintained: we merely solve the analytic problem of deriving the wavefunction that satisfies a quantum wave equation given this potential. This problem itself contains several surprising results; once these are described, we then briefly discuss the physical meaningfulness of such a potential. We also remark here that the Hamiltonian (1) is complex, which is significant and clearly requires some analysis, and we also discuss this feature. For the moment, we note that complex quantum Hamiltonians have previously been described [4]; this one has the merits of being very simple, and describing to lowest order \( n \) symmetric particles.
orbiting a center C at radius $r$ and speed $V = \omega r/k$. Most important, this simple Hamiltonian has the added virtue of possessing exact solutions that do not appear to have previously been described in the literature.

Let us start by deriving solutions to the 2-D Klein-Gordon equation [5] in the presence of the interaction Hamiltonian (1). We will show that, unexpectedly, there is an exact solution to this equation only if certain conditions on $P(r)$ and the expectation value of the angular momentum operator are met. The Klein-Gordon equation in 2D reads:

$$\nabla^2 \psi - \frac{\mu^2 c^2}{\hbar^2} \psi = -\frac{1}{\hbar^2 c^2} \left( \frac{\hbar}{\text{d}t} - H_1 \right)^2 \psi$$

$$= \frac{1}{c^2} \frac{\text{d}^2 \psi}{\text{d}t^2} + \frac{2i}{\hbar c^2} \frac{\text{d}p}{\text{d}t} + \frac{i}{\hbar c^2} \left( \frac{\text{d}H_1}{\text{d}t} \right) \psi - \frac{\left( H_1 \right)^2}{\hbar c^2} \psi,$$

and when we insert the Hamiltonian from Eq. (1), we find that the resulting equation is not separable by normal techniques. We can nevertheless proceed by attempting a solution of the following, doubly exponentiated, form:

$$\psi = M(r) \cdot \exp \left[ i(\hat{k}\theta - \hat{\omega}t) + f r^g \exp[i(\hat{k}\theta - \hat{\omega}t)] \right]$$

(3)

where $\hat{k}$, $\hat{\omega}$, $f$ and $g$ are constants and $M$ is an unknown function of radius alone. This allows us to expand Eq. (2) and collect terms in like powers of $\exp[i(\hat{k}\theta - \hat{\omega}t)]$:

$$0 = \left\{ M'' + \frac{1}{r} M' + \left[ \frac{\hat{\omega}^2}{r^2} - \frac{\mu^2 c^2}{\hbar^2} - \frac{k^2}{r^2} \right] M \right\}$$

$$+ \left[ 2fgM + \frac{1}{r} \left[ \frac{\hat{\omega}^2}{r^2} - \frac{k^2}{r^2} + \frac{2\hat{\omega}}{r} H_1 \right] \right] r^{g-1} \exp[i(\hat{k}\theta - \hat{\omega}t)]$$

$$+ \left[ \left( g^2 - k^2 \right) r^{2g-2} + \frac{\omega}{r} \frac{\text{d}g}{\text{d}r} \frac{\text{d}r}{\text{d}t} \right]$$

$$r^{2g-2} \exp[i(\hat{k}\theta - \hat{\omega}t)]$$

(4)

where the prime notation, $M'$, represents differentiation with respect to the radius, $r$. Each of the quantities in curly brackets must vanish independently, so we have transformed the PDE (2) into three simultaneous ODE’s. A solution can now exist only if all three ODE’s can be made consistent.

Up to this point, $P(r)$ has been arbitrary. We can deduce from the last curvy bracket in Eq. (4) that only one form of $P(r)$ will permit a solution. This last bracket contains several powers of $r$, all of which must cancel to permit a solution at all $r$. In particular, the bracket contains a term in $r^{2g}$, which can only vanish if $P(r) \sim r^g$. Likewise, the last bracket contains a term in one other fixed power: $r^{2g-2}$, which requires that $P(r)$ contain a term in $r^{g-1}$ as well, so:

$$P(r) = P_g r^g + P_{g-1} r^{g-1},$$

(5)
where $P_g$ and $P_{g-1}$ are undetermined coefficients. No other terms are possible.

Expanding the last bracket in Eq. (4), produces:

$$0 = \left( g^2 - k^2 + \frac{P_{g-1}^2}{f^2 h^2 c^2} \right) r^{2g-2} + \left( 2P_g P_{g-1} - \frac{2\omega P_{g-1}}{fhc^2} \right) r^{2g-1} + \left( \frac{\omega^2}{c^2} + \frac{P_g^2}{f^2 h^2 c^2} - \frac{2\omega P_g}{fhc^2} \right) r^g$$  \hspace{1cm} (6)

The problem here is overconstrained: we have three conditions in only two unknowns, $P_g/f$ and $P_{g-1}/f$. It is easily confirmed that the conditions can be satisfied in exactly one way:

$$\frac{P_{g-1}}{f} = \pm hc \sqrt{k^2 - g^2}$$  \hspace{1cm} (7)

$$\frac{P}{f} = \pm \hbar \omega.$$  \hspace{1cm} (8)

Thus of mathematical necessity, only the one form of attractive potential defined by Eqs (5), (7) and (8) can produce the exact solution (3). We can now apply conditions (7) and (8) to the rest of Eq. (4) and seek a power series solution for $M(r)$:

$$M = \sum M_m r^m$$  \hspace{1cm} (9)

The first and second curvy brackets in Eq. (4) then become:

$$0 = \left[ m^2 + 4m + 4 - k^2 \right] M_{m+2} + \left[ \frac{\omega^2}{c^2} - \frac{\hbar^2}{2} \right] M_m$$  \hspace{1cm} (10)

$$0 = \left[ m + 1 + \frac{1}{2g} \left( g^2 - k^2 - 2kk \right) \right] M_{m+1} - \left[ \frac{\omega^2 \omega_{g-1}}{2fghc^2} \right] M_m$$

$$+ \left[ \frac{1}{gc^2} \left( \omega^2 + \frac{\omega}{2} - \frac{2\omega P_g}{2fh} \right) \right] M_{m-1}$$  \hspace{1cm} (11)

Equations (10) and (11) can be simultaneously solved subject to conditions that can be derived in a straightforward manner. The first of these is that Eq. (11) can contain nonzero terms only in $M_{m+1}$ and $M_m$; it can be verified by substitution that the last term leads to incompatibilities with Eq. (10). Thus the last term in Eq. (11) must vanish, so:

$$P_g = \frac{2fh}{3\omega} \left( \omega^2 + \frac{\omega^2}{2} \right)$$  \hspace{1cm} (12)

This will be compatible with the earlier condition (8) iff:

$$\dot{\omega} = \omega.$$  \hspace{1cm} (13)
We can then iterate Eq. (11) once to obtain:

\[
0 = \left[ m^2 + \left( 3 + \frac{\sigma}{g} \right) m + \left( 1 + \frac{\sigma}{2g} \right) \left( 2 + \frac{\sigma}{2g} \right) \right] M_{m=2} - \left[ \frac{3\omega P_{g=1}}{2fghc^2} \right]^2 M_m, \tag{14}
\]

where \( \sigma = g^2 - k^2 - 2\hat{k} \). Comparing this with Eq. (10), we get three final consistency conditions;

\[
g^2 - g = k^2 + 2\hat{k} \tag{15}
\]

\[
\hat{k} = \pm \frac{1}{2} \tag{16}
\]

\[
P_{g=1} = \pm \frac{2fghc}{3} \eta \tag{17}
\]

where we have used the abbreviation:

\[
\eta = \sqrt{1 - \frac{\mu^2c^4}{\hbar^2\omega}}. \tag{18}\]

The solution subject to these conditions is then the power series (9) and its recursion relation (10). A little algebra condenses this to:

\[
\varphi = \frac{\varphi_0}{\sqrt{r}} \exp \left\{ \left( \pm \frac{\Theta}{2} - o\tau \right) \pm \frac{\omega_1}{\epsilon} r \pm fr \cdot \exp [i(k\theta - o\tau)] \right\} \tag{19}
\]

In order for Eq. (19) to be square-normalizable, we require \( g = 0 \). For the same reason, only stable states are of interest, so only negative real terms are allowed in the exponential, and consequently solutions with positive \( \pm \omega_1/\epsilon \) must be discarded.

Finally, the probability density, \( \varphi^* \varphi \), must be single valued. From examination of (19), this implies that the wave number, \( \hat{k} \), can only be an integer. By Eq's (15) and (16), we conclude that the candidate values for \( k \) are 0, +1, and -1, so at last the only consistent, square-normalizable solution is:

\[
P(r) = \pm \hbar \omega \tag{20}
\]

\[
\varphi = \frac{\varphi_0}{\sqrt{r}} \exp \left\{ \left( \pm \frac{\Theta}{2} - o\tau \right) - \frac{\omega_1}{\epsilon} r + f \cdot \exp [i(k\theta - o\tau)] \right\} \tag{21}
\]

where \( k = 0, \pm 1 \). The corresponding probability density for this wave function is:

\[
\varphi^* \varphi = \varphi_0^2 \frac{\exp(-kr)}{r} \exp\left[ 2f\cos(k\theta - o\tau) \right] \tag{22}
\]

where \( \kappa = 2\sqrt{\hbar^2\omega^2 - \mu^2c^4}/\hbar \) is a constant.
We can use this solution to compute a number of observables. First, the solution (21) has three free parameters: \( \kappa \), \( f \), and \( \omega \). The value of \( \kappa \) does not affect the qualitative character of the solution: it merely alters the strength of the Yukawa-like shielding term in Eq. (220). Likewise, the magnitude of \( \omega \) increases the total energy and rate of rotation of the wave function, but has no effect on a qualitative description of the solution. Only \( f \), the magnitude of the attractive potential, qualitatively alters the wave function. Two cases can be discriminated -- case 1: \( f = O(1) \), and case 2: \( f \ll 1 \). For each of these cases, the possible solutions come in three varieties, one for each of the allowed values of \( k \): \( k = -1, 0 \) or \( +1 \). The choice of \( k \) is independent of \( \langle L_z \rangle \), which is fixed. In Fig. 2, we plot the real part of the wave function for each allowed value of \( k \), and for each case, \( f = O(1) \), and \( f \ll 1 \).

For case 1, the real part of the wavefunction with \( k = 1 \) is shown in top and bottom views respectively in Fig's 1(a) and 1(d). As time evolves, the wavefunction in all cases and varieties orbits the origin, \((X,Y) = (0,0)\) with constant angular velocity, \( \omega \). The wavefunction for large \( f \) and \( k = 1 \) is surprisingly complex -- for example, its shape viewed from above has three, while from below it has only two. It is not clear whether this property would produce direct observables, but this type of behavior is curious and seems to be unique.

Again for case 1, the wavefunction with \( k = -1 \) is shown in top and bottom views respectively in Fig's 1(c) and 1(d). This wavefunction is again asymmetric above and below \( \text{Re}(\phi) = 0 \). Moreover, the wavefunction for both \( k = 1 \) and \( k = -1 \) lives almost entirely in only one half plane, here \( Y > 0 \). This
suggests that a copy of either the \( k = 1 \) or the \( k = -1 \) state could be placed in the other half plane with minimal interference from the second solution. We return to this issue shortly.

Finally, for either case 1 and \( k = 0 \), or case 2 (and any choice of \( k \)), the wavefunction takes on the appearance shown in Fig. 1(b). Here a constant \( \varphi \) cross section resembles a cycloid. Although these states orbit the origin as well, they are substantially more nearly azimuthally symmetric than the previously described states and may very well exhibit distinguishable observable behaviors as a result.

Beyond observables associated with the solution (21), a particular form for the radial interaction, \( P(r) \) is required. Using Eq. (20), the interaction (1) becomes:

\[
H_I(r,\theta,t) = \pm \hbar \omega \exp\left[\mathbf{i}(k\theta - \omega t)\right].
\]  

This interaction, and this interaction alone, admits a consistent solution of the form (3).

A second observable associated with the solution (21) is the angular momentum. We note that Eq. (21) contains a term in \( \exp\left[\pm i\vartheta/2\right] \), which implies that the expectation value of the orbital angular momentum operator, \( \mathbf{L}_z \), must \textit{identically} equal \( \pm \hbar/2 \). We emphasize that this is due to a \textit{mathematical} constraint (expressed in Eq. (16)), and is not based on any known physical principle. This suggests that the so-called ‘spin crisis’ may have a mathematical resolution, namely that the three varieties of exact solutions that we have derived each has \textit{orbital angular momentum of exactly \( \hbar/2 \)} – and so deep inelastic scattering measurements will of necessity reveal only this orbital angular momentum [6]. Only by orbiting a common center, in a field specified by \( H_I \), can these solutions exist, and the solutions contain no other, intrinsic, angular momentum.

We note that by the same token that a single exact solution may exist subject to the Hamiltonian, \( H_I \), this solution cannot exist in isolation: it requires that at least one other particle must exist to set up this Hamiltonian, and whatever particle this might be, it must generate a field that rotates with the same speed as the wave function. Thus the picture suggested by this analysis is that individual solutions, which seem consistent with existing descriptions of quarks, cannot exist in isolation, but require an attractive field set up by what we can only presume are produced by intermediary gluons. How, precisely, the gluons generate this orbiting Hamiltonian is not defined by the mathematics that we present: we only present the fact that they must do so to sustain these exact solutions.

The model that we have described does, however, provide us with a clue to a future understanding of how this might occur. As we have mentioned, complex Hamiltonians have been discussed previously in the literature, however a troublesome feature that they present is that the imaginary component would seem to make the Hamiltonian non-unitary. A careful calculation, however, reveals that the \textit{integrated} value for the total energy: \( E = \int_0^\infty \int_0^{2\pi} \int_0^{\hbar/\omega} H_I \, dr \, d\theta \, dt \) is conserved, where \( H_I^* \) is the adjoint. Thus the energy
fluctuates as the exact wavefunctions defined orbit the center, and these fluctuations orbit in phase with the solutions, indicating that if we interpret the solutions as quarks, then they must set up a gluon field that orbits with them. Likewise the Hamiltonian prescribes that as the solutions (21) orbit a common center, they locally lose energy periodically in time, with frequency $\omega$, in such a way that the global energy, $E$, is conserved. We will look to future analytic and experimental work to investigate the extent to which this exact solution may, or may not, contribute to an understanding of the spin crisis and related problems.

In conclusion, we have presented an exact solution to an $n$-fold azimuthally periodic Hamiltonian with three free parameters: a frequency, $\omega$, an energy, $\kappa$, and a dimensionless field strength, $f$. The solution describes a state with fixed angular momentum, $\hbar/2$, and finite radius that comes in three distinguishable varieties (Fig’s 2). Surprisingly, in order to generate such a state, there is no freedom for the radial form of the potential, for the wave number of the solution, or for the angular momentum of a bound state.
References:


3 For reviews, see:

4 cf. Regge, T., Nuovo Cim. 14 (1959);

5 We study the relativistic case for generality, although we have found a similar solution also in the non-relativistic, Schrödinger, limit.