Let $\phi(\tau) = \bar{\phi}(\alpha_1, \alpha_2, \alpha_3)$, where

$$\alpha_j = j^{-1} \text{tr} \tau^j, \quad j = 1, 2, 3. \quad (1)$$

We can use these three invariants instead of $\beta_1, \beta_2, \beta_3$ since $\phi$ is isotropic. The advantage is that the $\alpha_j$ are explicit functions of $\tau$, whereas the $\beta_A$ are not so explicit (they can be expressed as roots of a cubic etc. but that is a nightmare). We'll return to the $\beta_A$ at the end. Also, we'll ignore $q$ for simplicity.

So,

$$\frac{\partial \phi}{\partial \tau} = \sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial \tau}$$

$$= \sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \tau^{j-1} \quad (2)$$

Then use the spectral form of $\tau$ along with

$$\tau^{j-1} = \sum_{A=1}^{3} \beta_A^{j-1} n_A \otimes n_A \quad (3)$$

to get

$$\frac{\partial \phi}{\partial \tau} = \sum_{A=1}^{3} \left( \sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \beta_A^{j-1} \right) n_A \otimes n_A$$

$$= \sum_{A=1}^{3} \left( \sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial \beta_A} \right) n_A \otimes n_A$$

$$= \sum_{A=1}^{3} \frac{\partial \hat{\phi}}{\partial \beta_A} n_A \otimes n_A \quad (4)$$

The last bit follows from

$$\alpha_j = j^{-1} \sum_{A=1}^{3} \beta_A^j \quad \Rightarrow \quad \frac{\partial \alpha_j}{\partial \beta_A} = \beta_A^{j-1} \quad (5)$$