Integer spin particles necessarily produce half-integer angular momentum in a simple complex and periodic Hamiltonian

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Exact wave functions are derived from an azimuthally periodic Hamiltonian using both the Klein-Gordon and the Schrödinger equations. We show that, curiously, for both relativistic and non-relativistic equations, *integer* spin wave equations necessarily produce half-integer angular momentum in this potential. We find additionally that the higher energy, relativistic, solutions require an asymptotically free potential, while the lower energy, Schrödinger, solutions can exist in a potential that grows linearly with r. These are purely mathematical results, however we speculate on possible physical interpretations.

It has been known for nearly two decades that the angular momentum of the proton is not accounted for by the intrinsic spins of constituent quarks, and must presumably be associated with orbital angular momentum [1]. This 'spin crisis' [2] has prompted numerous theoretical, computational, and experimental investigations [3]. In this letter, we do not attempt to resolve the spin crisis *per se*; rather we consider the crisis as evidence that non-trivial angular momenta can appear in confined quantum states. Motivated by this observation, we calculate the simplest possible quantum solutions for *integer* spin particles in a central, but azimuthally varying, field. This has not, to our knowledge, been done before, and we show that this calculation yields only three possible varieties of exact solutions. Surprisingly these solutions can only exist for a particular form of central field, and only if the orbital angular momentum of the system is a *half*-integer, i.e. $\hbar/2$. No other solutions are possible. We derive this result, and then discuss possible implications.

We begin with a classical cartoon of the problem in Fig. 1. Here we consider a number, n, of discrete particles orbiting a common center, C, each with velocity V. One of the *simplest* Hamiltonians that could describe this system would be n-fold azimuthally periodic, i.e.:

$$H_I(r, \vartheta, t) = P(r) \exp[i(k\vartheta - \omega t)] , \qquad (1)$$

where k and ω are real constants, $k = 2\pi/n$, and P(r) is a radial interaction, which is to be determined. In this cartoon, we depart from the usual central potential formulation, in which an azimuthally symmetric potential confines particles (e.g. in the Bohr model). Instead, we ask what solutions exist for a potential that varies with ϑ so that particles orbiting the center experience an unchanging central force. We do not describe at this point how such a potential would be maintained: we merely derive the wavefunction that satisfies a quantum wave equation given this potential. This problem contains several surprising results; once these have been obtained, we will briefly discuss possible physical meanings of such a potential.

Figure 1 – Classical cartoon of n=3 particles orbiting a common center C, with velocity V.
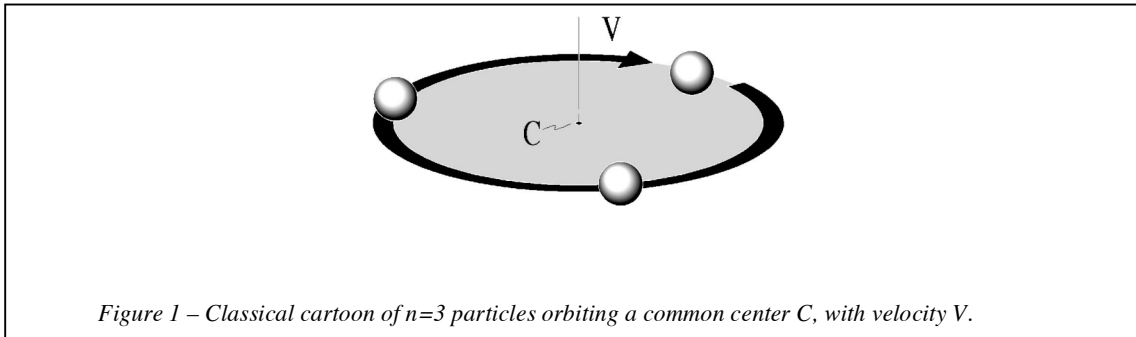
Before beginning, we remark that the Hamiltonian (1) is complex, which is significant and will be discussed shortly. For the time being, we note that complex quantum Hamiltonians have previously been described [4]; this one has the merits of being very simple, of describing to lowest order n particles orbiting a center C at radius r and speed $V = \omega r/k$, and, as we will see, of being exactly solvable.

We start by deriving solutions to the 2-D Klein-Gordon equation in the presence of the interaction Hamiltonian (1). We use the Klein-Gordon equation because this equation describes *integer* spin dynamics, and as we will show, unexpectedly the resulting *integer* spin wavefunction can only exist with *half*-integer orbital angular momentum. This, admittedly peculiar, result appears from purely mathematical considerations. Once we have obtained the more general, relativistic, Klein-Gordon result, we present the corresponding solution in the non-relativistic, Schrödinger, limit, which has its own unexpected properties. The Klein-Gordon equation in 2D reads:

$$\nabla^2\varphi \; - \; \frac{\mu^2 c^2}{\hbar^2}\varphi \;\; = \;\; \frac{1}{c^2}\frac{d^2\varphi}{dt^2} \; + \; \frac{2i}{\hbar c^2}H_I\frac{d\varphi}{dt} \; + \; \frac{i}{\hbar c^2}\left(\frac{dH_I}{dt}\right)\varphi \; - \; \left(\frac{H_I}{\hbar c}\right)^2\varphi\,, \tag{2}$$

and when we insert the Hamiltonian from Eq. (1), we find that the resulting equation is not separable by normal techniques. We can proceed by attempting a solution of the doubly exponentiated form:

$$\varphi \;\; = \;\; M(r)\cdot\exp\left\{i(\hat{k}\vartheta - \hat{\omega}t) + fr^g\exp[i(k\vartheta - \omega t)]\right\} \tag{3}$$

where $\hat{k}$, $\hat{\omega}$, f and g are constants and M is an unknown function of radius alone. This allows us to expand Eq. (2) and collect terms in like powers of $\exp[i(k\vartheta - \omega t)]$:

$$
\begin{aligned}
0 \;\; = \;\; & \left\{M'' + \frac{1}{r}M' + \left[\frac{\hat{\omega}^2}{c^2} - \frac{\mu^2 c^2}{h^2} - \frac{\hat{k}^2}{r^2}\right]M\right\} \\
& + \left\{2fgM' + fM\left[\frac{g^2 - k^2 - 2k\hat{k}}{r} + \left(\frac{2\omega\hat{\omega} + \omega^2}{c^2}\right)r - \frac{3\hat{\omega}P(r)r^{1-g}}{f\hbar c^2}\right]\right\}r^{g-1}\exp[i(k\vartheta - \omega t)] \\
& + \left\{\left[(g^2 - k^2)r^{2g-2} + \frac{\omega^2}{c^2}r^{2g} - \frac{2\omega}{f\hbar c^2}P(r)r^g + \frac{1}{f^2\hbar^2 c^2}P^2(r)\right]M\cdot\exp[2i(k\vartheta - \omega t)]\right\}
\end{aligned}
\tag{4}
$$

where the prime notation, M', represents differentiation with respect to the radius, r. Each of the quantities in curly brackets must vanish independently, so we have transformed the PDE (2) into three simultaneous ODE's. A solution can now exist only if all three ODE's can be made consistent.

Up to this point, $P(r)$ has been arbitrary. We can deduce from the last curvy bracket in Eq. (4) that only one form of $P(r)$ will permit a solution. This last bracket contains several powers of r, all of which must cancel to permit a solution at all r. In particular, the bracket contains a single term in r^{2g}, which can only vanish if $P(r) \sim r^g$. Likewise, the last bracket contains a term in one other fixed power: r^{2g-2}, which requires that $P(r)$ contain a term in r^{g-1} as well, so:

$$P(r) \;\; = \;\; P_g r^g \; + \; P_{g-1}r^{g-1}\,, \tag{5}$$

where P_g, and P_{g-1} are undetermined coefficients. No other terms are possible. Expanding the last bracket in Eq. (4), produces:

$$0 = \left(g^2 - k^2 + \frac{P_{g-1}^2}{f^2\hbar^2 c^2}\right)r^{2g-2} + \left(\frac{2P_g P_{g-1}}{f^2\hbar^2 c^2} - \frac{2\omega P_{g-1}}{f\hbar c^2}\right)r^{2g-1} + \left(\frac{\omega^2}{c^2} + \frac{P_g^2}{f^2\hbar^2 c^2} - \frac{2\omega P_g}{f\hbar c^2}\right)r^{2g} \qquad (6)$$

The problem here is overconstrained: we have three conditions in only two unknowns, P_g/f and P_{g-1}/f. It is easily confirmed that the conditions can be satisfied in exactly one way:

$$\frac{P_{g-1}}{f} = \pm\hbar c\sqrt{k^2 - g^2} \qquad (7)$$

$$\frac{P_g}{f} = \hbar\omega . \qquad (8)$$

Thus of mathematical necessity, only the one form of attractive potential defined by Eq's (5), (7) and (8) can produce the exact solution (3). We can now apply conditions (7) and (8) to the rest of Eq. (4) and seek a power series solution for M(r):

$$M = \sum M_m r^m \qquad (9)$$

The first and second curvy brackets in Eq. (4) then become:

$$0 = \left[m^2 + 4m + 4 - \hat{k}^2\right]M_{m+2} + \left[\frac{\hat{\omega}^2}{c^2} - \frac{\mu^2 c^2}{\hbar^2}\right]M_m \qquad (10)$$

$$0 = \left[g(m+1) + \frac{1}{2}\left(g^2 - k^2 - 2k\hat{k}\right)\right]M_{m+1} - \left[\frac{3\hat{\omega}P_{g-1}}{2f\hbar c^2}\right]M_m + \left[\frac{1}{c^2}\left(\omega\hat{\omega} + \frac{\omega^2}{2} - \frac{3\hat{\omega}P_g}{2f\hbar}\right)\right]M_{m-1} \qquad (11)$$

Equations (10) and (11) can be simultaneously solved subject to conditions that can be derived in a straightforward manner. The first of these is that Eq. (11) can contain nonzero terms only in M_{m+1} and M_m: it can be verified by substitution that the last term leads to incompatibilities with Eq. (10). Thus the last term in Eq. (11) must vanish, so:

$$P_g = \frac{2f\hbar}{3\hat{\omega}}\left(\omega\hat{\omega} + \frac{\omega^2}{2}\right) \qquad (12)$$

This will be compatible with the earlier condition (8) if and only if $\hat{\omega} = \omega$. We can then iterate Eq. (11) once to obtain:

$$0 = \left[g^2 m^2 + (3g + \sigma)gm + \left(g + \frac{\sigma}{2}\right)\left(2g + \frac{\sigma}{2}\right)\right]M_{m+2} - \left[\frac{3\hat{\omega}P_{g-1}}{2f\hbar c^2}\right]^2 M_m , \qquad (13)$$

where $\sigma = g^2 - k^2 - 2k\hat{k}$. Comparing this with Eq. (10), we get three final consistency conditions;

$$g^2 - g = k^2 + 2k\hat{k} \qquad (14)$$

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$$\hat{k} \; = \; \pm \frac{1}{2} \tag{15}$$

$$P_{g-1} \; = \; \pm \frac{2fg\hbar c}{3}\eta \tag{16}$$

where we have used the abbreviation $\eta \; = \; \sqrt{\hbar^2\omega^2 \, - \, \mu^2 c^4}\big/\hbar\omega$. The solution subject to these conditions is then the power series (9) and its recursion relation (10). A little algebra condenses this to:

$$\varphi \; = \; \frac{\varphi_o}{\sqrt{r}}\exp\left\{i\left(\pm\frac{\vartheta}{2} - \omega t\right) \pm \frac{\omega\eta}{c}r \pm fr^g \exp[\,i(k\vartheta - \omega t)]\right\}. \tag{17}$$

In order for Eq. (17) to be square-normalizable, we require $g = 0$, and $\pm\omega\eta/c < 0$. Finally, the probability density, $\varphi^*\varphi$, must be single valued. From examination of (18), this implies that the wave number, k, can only be an integer. By Eq's (14) and (15), we conclude that the candidate values for k are 0, +1, and -1, <u>so at last the only consistent, square-normalizable solution is:</u>

$$P(r) \; = \; f\hbar\omega \tag{18}$$

$$\varphi \; = \; \frac{\varphi_o}{\sqrt{r}}\exp\left\{i\left(\pm\frac{\vartheta}{2} - \omega t\right) - \left|\frac{\omega\eta}{c}\right|r \; + \; f\cdot\exp\left[i\left(k\vartheta - \omega t\right)\right]\right\} \tag{19}$$

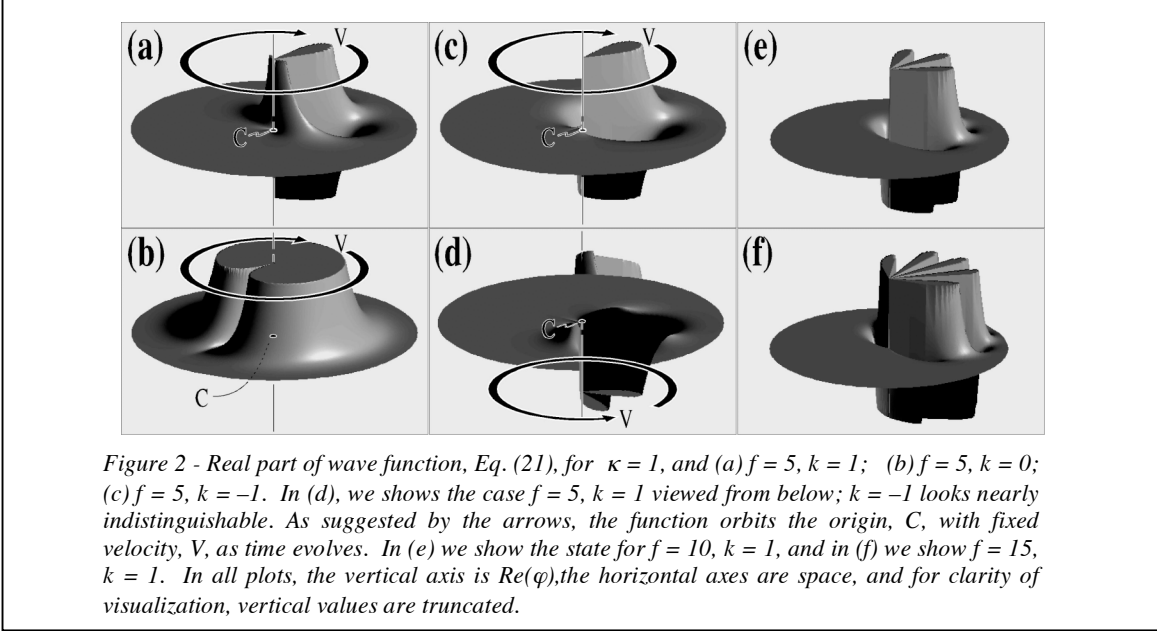where $k = 0, \pm 1$. The corresponding probability density for this wave function is:

$$\varphi^*\varphi \; = \; \varphi_o^2\frac{\exp(-\kappa r)}{r}\exp\left[2f\cos\left(k\vartheta - \omega t\right)\right] \tag{20}$$

where $\kappa \; = \; 2\sqrt{\hbar^2\omega^2 \, - \, \mu^2 c^4}\big/\hbar c$ is a constant.

We can use this solution to compute a number of observables. First, the solution (19) has three free parameters: κ, f, and ω. The value of κ does not affect the qualitative character of the solution: it merely alters the strength of the Yukawa-like shielding term in Eq. (20). Likewise, the magnitude of ω increases the total energy and rate of rotation of the wave function, but has no qualitative effect on the solution. Only f, the magnitude of the attractive potential, qualitatively alters the wave function. In Fig. 2, we plot the real part of the wave function for each allowed value of k. In Fig. 2(a)-(d), we have chosen f = 5 for illustrative purposes. For smaller f, the state becomes more azimuthally symmetric, whereas for larger f, the number of lobes grows: in Fig's 2(e)-(f), we display $k = \kappa = 1$ states for f = 10 and 1.

For nonzero f, the possible solutions come in three varieties, one for each allowed value of k: k = -1, 0 or +1. In the first variety, the real part of the wavefunction with k = 1 is shown in top and bottom views respectively in Fig's 2(a) and (d). In all cases, the wavefunction orbits the origin with constant angular velocity, ω. The wavefunction for large f and k = 1 is somewhat complex – its shape viewed from above in Fig. 2(a) has three lobes, while from below in Fig. 2(d) it has only two.

In the second variety of wavefunction, k = -1, the state is shown in Fig's 1(c) and (d). The wavefunction looks identical from below to the k = 1 state, but has fewer lobes when viewed from above. In the final variety, where k = 0, the wavefunction is shown in Fig. 1(b). Here a constant φ cross section resembles a cycloid. Although these states orbit the origin as well, the shape of this wavefunction is much more azimuthally symmetric than the previously described states and as a result should be expected to produce more symmetric elastic scattering patterns than the k = ±1 states.

Figure 2 - *Real part of wave function, Eq. (21), for κ = 1, and (a) f = 5, k = 1; (b) f = 5, k = 0; (c) f = 5, k = –1. In (d), we shows the case f = 5, k = 1 viewed from below; k = –1 looks nearly indistinguishable. As suggested by the arrows, the function orbits the origin, C, with fixed velocity, V, as time evolves. In (e) we show the state for f = 10, k = 1, and in (f) we show f = 15, k = 1. In all plots, the vertical axis is Re(φ), the horizontal axes are space, and for clarity of visualization, vertical values are truncated.*

Beyond observables associated with the solution (19), a particular form for the radial interaction, P(r) is required. Using Eq. (18), the interaction (1) becomes:

$$H_I(r,\vartheta,t) = f\hbar\omega \cdot \exp\left[i\left(k\vartheta - \omega t\right)\right].$$ (21)

This interaction, and *this interaction alone*, admits a consistent solution of the form (3).

A second observable associated with the solution (19) is the angular momentum. Eq. (19) contains a term in $\exp\left[\pm i\vartheta/2\right]$, where the constant ½ is obtained in Eq. (15), and is the only admissible value for this term: this is a consequence of the required consistency between the three ODE's, Eq's (10), (11) and (13). The value of this constant implies that <u>the angular momentum, $\langle L_z \rangle$, must *identically* equal ± $\pm \hbar/2$</u>. We stress that this is a *mathematical* constraint, and is not based on any known physical principle.

We return to this point shortly; first it is worthwhile to compare the solution obtained for the relativistic, Klein-Gordon, equation with the analogous solution for the non-relativistic, Schrödinger, equation. By following precisely the same form of derivation as has been presented above, one arrives at the following solution to the Schrödinger equation. The reader can verify that there are two classes of solutions and that all possible Schrödinger solutions also require that $\langle L_z \rangle$ must identically equal ± $\pm \hbar/2$.

The first class of solutions is essentially identical to the Klein-Gordon solution, with $2\mu c^2 / \hbar\omega$ substituted for η. The second class of solutions is more interesting, and is:

$$\varphi = \frac{\varphi_0}{\sqrt{r}} \left\{ \exp\left[i\left(\pm \frac{\vartheta}{2} + \hat{\omega}t \right) - \sqrt{\frac{2\mu\hat{\omega}}{\hbar}} \cdot r + \hat{f} \cdot r \cdot \exp\left[i(k\vartheta - \omega t) \right] \right] \right\} \quad (22)$$

$$H_I = -\hat{f}\hbar \left\{ \sqrt{\frac{2\hbar\hat{\omega}}{\mu}} - \frac{\hbar k(\pm 1 - k)}{2\mu r} + \omega \cdot r \right\} \cdot \exp\left[i(k\vartheta - \omega t) \right] \quad (23)$$

where $\hat{f}$, k, $\hat{\omega}$ and ω are parameters subject to the constraints $\hat{f}(k^2 - 1) = 0$ and $\hat{f}^2 < 2\hat{\omega}\mu / \hbar$. This second class of Schrödinger solutions defines spin ½ particles that live in a potential with three possible terms. The 1st term in braces in Eq. (23) is radially-independent, and closely resembles the relativistic Hamiltonian (21). The 2nd term is a simple $1/r$ potential that may or may not be present, depending on the choice of k. The 3rd term is the most noteworthy, for it defines a <u>potential that grows linearly with radius, r.</u>

Thus the treatment that we have presented yields to our knowledge the first derivation from first principles of a growth in potential with r, as is required to produce quark confinement. We may speculate on the meaning of this finding, however irrespective of physical interpretation, the mathematical results that we have presented are rigorous and cannot be adjusted. At high energies, the <u>only</u> permissible solution using Hamiltonian (1) is radially-independent (eq. (21)), whereas at low energies, this Hamiltonian can be supplemented by both $1/r$- and r- dependent potentials.

We note that by the same token that the exact solutions derived may exist subject to the Hamiltonians (21) and (23), these solutions <u>cannot exist in isolation</u>: at least one other particle must be present to set up the Hamiltonian, and in some way this particle or particles must generate a field that rotates with the wave function. How, precisely, this may occur is not defined by the mathematics that we present: we only present the fact that this must occur to sustain these exact solutions.

The model that we have described does, however, provide us with a lead to guide future analysis. As we have mentioned, complex Hamiltonians have been discussed previously in the literature, however a troublesome feature that they present is that the imaginary component would seem to make the Hamiltonian non-unitary. A careful calculation, however, reveals that the <u>integrated</u> value for the total energy: $E = \int_0^\infty \int_0^{2\pi} \int_0^{1/\omega} H_I^* H_I \; dr \; d\vartheta \; dt$ is conserved, where H_I^* is the adjoint. Thus the energy fluctuates as the wavefunctions defined orbit the center, and these fluctuations orbit in phase with the solutions, indicating that if we interpret the solutions as quarks, then they must set up a gluon field that orbits with them. Likewise the Hamiltonians (21) and (23) prescribe that as the solutions (19) and (22) orbit the origin, they locally lose energy periodically in time, with frequencies ω and $\hat{\omega}$, in such a way that the global energy, E, is conserved. We will look to future analytic and experimental work to investigate the extent to which this exact solution may, or may not, contribute to an understanding of the spin crisis and related problems.

In conclusion, we have presented exact solutions to an azimuthally periodic Hamiltonian. All solutions have fixed angular momentum, $\hbar/2$, and come in three distinguishable varieties (Fig's 2). Surprisingly, in the relativistic limit, in order to generate these exact solutions, there is <u>no</u> freedom for the radial form of the potential, for the wave number of the solution, or for the angular momentum of the bound state. Equally surprisingly, in the non-relativistic limit, a similar solution can exist in a potential that grows linearly with radius. These results are strictly mathematical in nature and are not subject to ambiguity. Their physical meaning, on the other hand, is considerably less clear.

From one perspective, the solutions that we have derived are satisfying insofar as they lend themselves to an unusual possible interpretation of the term 'intrinsic' angular momentum. That is, we speculate that spin ½ states may have this angular momentum not because a particle itself is spinning in some way, but because the only mathematically admissible wavefunctions define particles that orbit a point with fixed angular momentum $\hbar/2$. Additionally, comparison between the Klein-Gordon and Schrödinger solutions suggests a possible analytical source of the coexisting phenomena of asymptotic freedom and confinement, namely that at small scales (and so at high energy), equilibrium bound states can only exist with a potential that is independent of distance, while at larger scales, the potential may – depending on parameter choice – grow linearly with distance.

From another perspective, beyond these, speculative, interpretations, our solutions raise at least as many questions as they resolve. Among these are the following.

1) We have analyzed the angular momentum of an individual particle in the Hamiltonian (1), but we have noted that the corresponding solution cannot exist in isolation. Can other orbiting particles set up periodic complex potentials as we have presumed?

2) Our analysis uses integer spin wave equations. Does the Dirac equation for half-integer spin have similar solutions? Can the approach we have described specify allowable potentials for bag models, which currently can be chosen only phenomenologically [5,6]?

3) The solution we have obtained assumes an equilibrium form of Hamiltonian (1) in which particles orbit in circular trajectories: might solutions be distorted into more complex, nonequilibrium, trajectories during a collision [7]?

We look to future investigations to address these new issues.

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