Let $\phi(\boldsymbol{\tau}) = \bar{\phi}(\alpha_1, \alpha_2, \alpha_3)$, where

$$\alpha_j = j^{-1} \operatorname{tr} \boldsymbol{\tau}^j, \qquad j = 1, 2, 3. \tag{1}$$

We can use these three invariants instead of $\beta_1, \beta_2, \beta_3$ since ϕ is isotropic. The advantage is that the α_j are explicit functions of $\boldsymbol{\tau}$, whereas the β_A are not so explicit (they can be expressed as roots of a cubic etc. but that is a nightmare). We'll return to the β_A at the end. Also, we'll ignore $\mathbf{q}$ for simplicity.

So,

$$\begin{aligned} \frac{\partial \phi}{\partial \boldsymbol{\tau}} &= \sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial \boldsymbol{\tau}} \\ &= \sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \boldsymbol{\tau}^{j-1} \end{aligned} \tag{2}$$

Then use the spectral form of $\boldsymbol{\tau}$ along with

$$\boldsymbol{\tau}^{j-1} = \sum_{A=1}^{3} \beta_A^{j-1} \mathbf{n}_A \otimes \mathbf{n}_A \tag{3}$$

to get

$$\begin{aligned} \frac{\partial \phi}{\partial \boldsymbol{\tau}} &= \sum_{A=1}^{3} \Big(\sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \beta_A^{j-1} \Big) \mathbf{n}_A \otimes \mathbf{n}_A \\ &= \sum_{A=1}^{3} \Big(\sum_{j=1}^{3} \frac{\partial \bar{\phi}}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial \beta_A} \Big) \mathbf{n}_A \otimes \mathbf{n}_A \\ &= \sum_{A=1}^{3} \frac{\partial \hat{\phi}}{\partial \beta_A} \mathbf{n}_A \otimes \mathbf{n}_A \end{aligned} \tag{4}$$

The last bit follows from

$$\alpha_j = j^{-1} \sum_{A=1}^{3} \beta_A^j \qquad \Rightarrow \qquad \frac{\partial \alpha_j}{\partial \beta_A} = \beta_A^{j-1} \tag{5}$$