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# Macroscopic traffic flow propagation stability for adaptive cruise controlled vehicles

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**Abstract**

Traffic flow propagation stability is concerned about whether a traffic flow perturbation will propagate and form a traffic shockwave. In this paper, we discuss a general approach to the macroscopic traffic flow propagation stability for adaptive cruise controlled (ACC) vehicles. We present a macroscopic model with velocity saturation for traffic flow in which each individual vehicle is controlled by an adaptive cruise control spacing policy. A nonlinear traffic flow stability criterion is investigated using a wavefront expansion technique. Quantitative relationships between traffic flow stability and model parameters (such as traffic flow and speed, etc.) are derived for a generalized ACC traffic flow model. The newly derived stability results are in agreement with previously derived results that were obtained using both microscopic and macroscopic models with a constant time headway (CTH) policy. Moreover, the stability results derived in this paper provide sufficient and necessary conditions for ACC traffic flow stability and can be used to design other ACC spacing policies.


*Keywords:* Traffic propagation stability; Adaptive cruise control (ACC); Traffic flow; Wavefront expansion; Intelligent transportation systems (ITS)

## 1. Introduction

Traffic flow stability is an important subject because congestion caused by an unstable traffic stream or traffic shockwave degrades the performance of road transportation networks. There is a large body of research that deals with stability of manual traffic flows (Whitham, 1974; Holland, 1998; Zhang, 1999; del Castillo, 2001; Yi et al., 2003). Some of these works are based on well-known microscopic manual driving car-following models (Holland, 1998; del Castillo, 2001) and others used aggregated macroscopic traffic flow models (Whitham, 1974; Zhang, 1999; Yi et al., 2003). Recently, a new class of so-called adaptive cruise control (ACC), or autonomous intelligent cruise control (AICC) system has been developed for vehicles operating in manual traffic, which are able not only to maintain a constant velocity but also, in addition, either a

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constant distance or a constant headway from the preceding vehicle. ACC systems have recently been equipped on some passenger vehicles in the market. As a consequence, the stability of traffic flow, when vehicles are operating under ACC, has received increasing attention by the automated highway systems (AHS) and intelligent transportation systems (ITS) community, and this is the topic of this paper.

Swaroop and Rajagopal (1999) first studied stability of traffic flow under an ACC spacing policy using an aggregated macroscopic traffic flow model for an open stretch highway, focusing on effect of different adaptive cruise control policies on traffic flow dynamics. Using a linearized stability analysis, they showed that the traffic flow equilibrium state was marginally stable under a constant time headway (CTH) policy; while the stability analysis of a spatially discretized system showed that the system was unstable under the CTH policy. The conclusions in Li and Shrivastava (2002) seemed to contradict the results in Swaroop and Rajagopal (1999) at a first glimpse. Instead of studying an open stretch highway, Li and Shrivastava (2002) studied a circular highway, in order to eliminate the entry and exit effects on the intrinsic stability property of the ACC CTH policy. Analyses based on a microscopic model, a spatially discrete model, and a spatially continuous model were discussed and several stability conclusions were obtained depending on the choice of aggregating biasing strategy that was used in abstracting the highway's macroscopic dynamics. Li and Shrivastava (2002) also concluded that a particular entry and exit traffic policy can destroy the intrinsic stability property of an open stretch highway system under a CTH policy. The ACC system dynamics were not considered in Li and Shrivastava (2002) and the authors claimed that the stability property of traffic flow should be robust to such unmodeled dynamics. Recently, Wang and Rajamani (2004) discussed and resolved the mathematical controversy between Swaroop and Rajagopal (1999) and Li and Shrivastava (2002), using a spatially discrete model, without considering the ACC dynamics. The results in Wang and Rajamani (2004) provided an explanation of the discrepancy between the results in Swaroop and Rajagopal (1999) and Li and Shrivastava (2002). Moreover, a concept of *unconditional traffic flow stability* was proposed and a sufficient condition was found to guarantee such stability under ACC spacing policies. According to Wang and Rajamani (2004), alternative spacing policies with superior stability properties should be designed instead of using the CTH policy for ACC vehicle systems.

In this paper, we discuss a framework to study the traffic flow stability using a general continuous traffic flow model for vehicles under ACC spacing policies. We propose to use a concept of traffic flow propagation stability to study the traffic flow characteristic. With the observation that traffic flow shockwave could result in traffic jam or unstable, the traffic flow propagation is concerned about the conditions under which a traffic perturbation will propagate and form a shockwave, i.e. discontinuous traffic flow or velocity. The macroscopic model proposed in this paper will represent both the spatially biasing strategy and the ACC vehicle dynamics. A traffic velocity saturation model is also introduced. An intrinsic stability criterion of the ACC traffic flow systems is presented using a wavefront expansion technique (Whitham, 1974; Yi et al., 2003). The stability criterion is independent of the particular ACC policy used, and the stability conditions derived are both necessary and sufficient. The stability results for the CTH policy are investigated and found consistent with all previous results obtained in Swaroop and Rajagopal (1999), Li and Shrivastava (2002) and Wang and Rajamani (2004). One attractive property of the proposed stability analysis technique is that it can be used to design and validate other ACC policies with more precision than prior techniques.

The rest of the paper is organized as follows. In Section 2, a macroscopic ACC traffic flow model is presented and a definition of propagation stability of traffic flow is presented. The main results are presented in Section 3 and traffic stability criteria are analyzed using a wavefront expansion technique. Comparisons and discussion for the CTH policy are presented in Section 4. Concluding remarks are presented in Section 5.

## 2. Macroscopic ACC traffic flow model and stability

We consider only a one lane highway without any on- or off-ramps.[1] We assume that all vehicles on the highway area equipped with the same adaptive cruise control strategy. Let $\rho(x, t)$ denote the highway density,

[1] The inlet and outlet flows are considered as the on- and off-ramps. We can drop this assumption by adding relaxation terms in right hand side of the conservation laws.

$q(x,t)$ the flow rate and $v(x,t)$ the traffic velocity at position $x$ along the highway at time $t$, respectively. By definition, $q(x,t) = \rho(x,t)v(x,t)$. Note that the traffic velocity $v(x,t)$ is the average or aggregated speed of all individual vehicles on the highway around $x$. Conservation of vehicles on the highway gives us the following equation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{1}$$

and the velocity dynamics can be written, in a general form, as

$$v(x,t+T) = \begin{cases} v_{\mathrm{f}} & \text{if } 0 \leqslant \rho(x+\Delta(x,t),t) \leqslant \rho_{\min}, \\ h(\rho(x+\Delta(x,t),t)) & \text{if } \rho_{\min} \leqslant \rho(x+\Delta(x,t),t) \leqslant \rho_{\max}, \end{cases} \tag{2}$$

where $T$ is the relaxation time of the ACC system dynamics, $v_{\mathrm{f}}$ is the highway free flow velocity, $\Delta(x,t)$ is the spatially biasing strategy distance, $\rho_{\min} = h^{-1}(v_{\mathrm{f}})$ and $\rho_{\max}$ is the maximum highway flow.[2] Since we consider whether and how the traffic perturbation propagates and forms a shockwave, it is appropriately assumed that the traffic flow variables $\rho(x,t)$, $q(x,t)$ and $v(x,t)$ are piecewise differentiable (before the shockwave is formed).

The underlying physical meaning given by Eq. (2) is that the speed of the traffic flow at time $t+T$ is regulated as the traffic concentration policy $h(\rho)$ at position $x+\Delta$ at the time $t$. This abstracting idea is inspired from a similar micro-mesoscopic traffic model derivation given in Zhang (1998). The distance $\Delta$ is not necessarily equal to the distance between the vehicle and the vehicle ahead of it. It could be some distance away and, in this case, traffic information could be broadcasted through the roadside devices or from vehicle to vehicle via wireless communication. In this paper, we use the same convention as in Li and Shrivastava (2002) to characterize the spatial biasing strategy. If $\Delta(x,t) > 0$, we will denote the spatial biasing as being a downstream biasing; if $\Delta(x,t) = 0$, it is a neutral biasing; and if $\Delta(x,t) < 0$, it is a upstream biasing. If we consider an ideal ACC system as in Li and Shrivastava (2002), then $T = 0$. In Swaroop and Rajagopal (1999), a first-order feedback control is assumed to regulate the vehicle velocity around $h(\rho)$ with a decaying time constant $\tau$. In Swaroop and Rajagopal (1999), only the distance between two vehicles is considered for the traffic control and we consider such an ACC spacing policy as having a neutral biasing, i.e. $\Delta(x,t) = 0$.

The spacing policy $h(\rho)$ is similar to the velocity–density fundamental diagram ($v$–$\rho$ curve) of manual traffic flow. We consider a general function form $h(\rho)$ of this policy. As in the $v$–$\rho$ fundamental diagram for manual traffic flow, the spacing policy function $h(\rho)$ of ACC traffic flow satisfies

$$h'(\rho) = \frac{\mathrm{d}h}{\mathrm{d}\rho} \leqslant 0. \tag{3}$$

For example, under the CTH policy, $h(\rho)$ is given by

$$h(\rho) = \begin{cases} v_{\mathrm{f}} & \text{if } 0 \leqslant \rho(x,t) \leqslant \rho_{\min}, \\ \frac{1}{h_{\mathrm{w}}}\left(\frac{1}{\rho} - L_{\mathrm{v}}\right) & \text{if } \rho_{\min} < \rho(x,t) \leqslant \rho_{\max}, \end{cases} \tag{4}$$

where $h_{\mathrm{w}}$ is the constant time headway and $L_{\mathrm{v}}$ is the average vehicle length. In this case, $\rho_{\min} = \frac{1}{v_{\mathrm{f}}h_{\mathrm{w}}+L_{\mathrm{v}}}$ and $\rho_{\max} = \frac{1}{L_{\mathrm{v}}}$. A schematic of traffic velocity and flow rate under the CTH policy is shown in Fig. 1. The above CTH policy (4) was used in Swaroop and Rajagopal (1999), Li and Shrivastava (2002) and Wang and Rajamani (2004).

Using Taylor's expansion and neglecting higher-order terms, from Eq. (2), we obtain

$$v(x,t) + \frac{\partial v}{\partial t}T = h(\rho(x,t)) + \Delta h'(\rho)\frac{\partial\rho}{\partial x},$$

and arrange the above equation as follows:

$$\frac{\partial v}{\partial t} - \mu h'(\rho)\frac{\partial\rho}{\partial x} = -\frac{1}{T}(v - h(\rho)), \tag{5}$$

[2] The highway maximum flow density can be simply calculated by $\frac{1}{L_{\mathrm{v}}}$, where $L_{\mathrm{v}}$ is the average vehicle length, i.e. the *bumper-to-bumper* flow density.

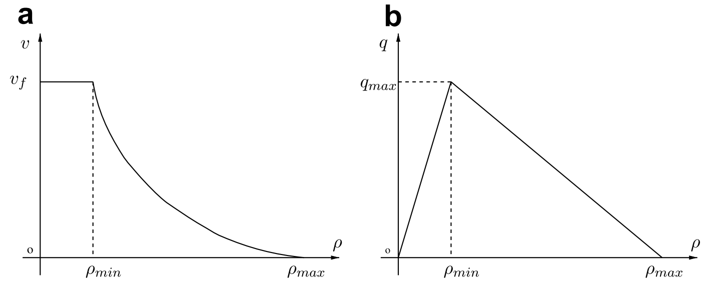


Fig. 1. A schematic of the CTH policy: (a) velocity–density relationship ($v$–$\rho$ curve); (b) flow–density relationship ($q$–$\rho$ curve).

where $\mu := \frac{\Delta(x,t)}{T}$. In this paper, we consider $\mu$ is a constant for simplicity. There are two types of coordinate systems for referencing the trajectory of the moving vehicle stream (Zhang, 1998; Swaroop and Rajagopal, 1999). One is the Euler coordinate system, in which the origin is a fixed point along the highway, and the other is Lagrangian coordinate system, in which the origin is fixed with the moving vehicle. Using different coordinate systems will produce different dynamical equations. In this paper, we use the Euler coordinate system, as in Zhang (1998), based on the observation that the vehicles could be controlled by roadside devices such as variable message signs etc. In Swaroop and Rajagopal (1999), the traffic velocity dynamics is abstracted from controlling the acceleration of each vehicle and using a Lagrangian coordinate system. Even though the traffic velocity dynamics in Swaroop and Rajagopal (1999) is different from Eq. (5), the stability conditions can still be obtained from the wavefront expansion technique proposed in this paper. We will discuss this in detail in Section 4.

We consider the ACC traffic flow dynamics given by Eqs. (1) and (5). In this paper, we discuss the propagation stability of ACC traffic flow under perturbed initial conditions. A formal definition of traffic flow stability (in the sense of Lyaponuv) was introduced in Swaroop and Rajagopal (1999) using the definition of stability for infinite-dimensional continuous dynamical systems.

**Definition 1** (*Traffic flow stability in Swaroop and Rajagopal (1999)*). Let $\mathbf{q}_e(x,t) = [\rho_e(x,t), v_e(x,t)]^{\mathrm{T}}$ denote the nominal equilibrium state of the traffic system on a highway with length $L$. Let $\mathbf{q}_p(x,t)$ be the perturbed state. The traffic flow $\mathbf{q}_e(x,t)$ is **stable** if

1. for a given $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\sup_{x \leqslant L}\{\|\mathbf{q}_p(x,0)\|\} < \delta \Rightarrow \sup_{t \geqslant 0}\sup_{x \leqslant L}\{\|\mathbf{q}_p(x,t)s\|\} < \epsilon.$$

2. $\lim_{t\to\infty}\sup_{x\leqslant L}\{\|\mathbf{q}_p(x,t)\|\} = 0$.

The definition of the traffic flow propagation stability, which this paper focus on, is given as follows (Yi et al., 2003).

**Definition 2** (*Traffic flow propagation stability*). Let $\mathbf{q}_e(x,t) = [\rho_e(x,t), v_e(x,t)]^{\mathrm{T}}$ denote the nominal equilibrium state of the traffic system on a highway with length $L$. Let $\mathbf{q}_p(x,t)$ be the perturbed state. The traffic flow $\mathbf{q}_e(x,t)$ is **propagation stable** under perturbation traffic state $\mathbf{q}_p$ if the spatial gradient of the perturbed state is bounded, i.e. $\|\frac{\partial \mathbf{q}_p}{\partial x}(x,t)\| < \infty$, for $\forall t > 0$, $x \in [0, L]$. If, in addition to the above, $\lim_{t\to\infty}\|\frac{\partial \mathbf{q}_p}{\partial x}(x,t)\| = 0$, then, the traffic state $\mathbf{q}_e$ is **asymptotically propagation stable**.

We prefer to use the definition of the traffic propagation stability because normally transportation engineers refer to traffic instability as the formation of a sustained shockwave or bottleneck (Yi et al., 2003). It is also observed from the above definitions that if the traffic flow is propagation unstable (by Definition 2), it must be unstable (by Definition 1). If the traffic flow is steady, the asymptotic propagation stability will also imply the traffic flow stability. Because of such relationships between these two definitions, we will concentrate on the propagation stability conditions and compare with existing results in the literature.

For a system governed by partial differential equations (PDE), if we consider that a perturbation happens on a smooth density or velocity profile and if the perturbation attenuates to the equilibrium state, we normally consider the system to be asymptotically stable. Therefore, if the traffic flow system satisfies the propagation stability and in addition $\left\|\frac{\partial q_p}{\partial x}\right\| \to 0$ as $t \to \infty$, we will consider that the system is asymptotically stable.

The wavefront of a traffic system can be illustrated by Fig. 2 as a separation curve between disturbed and undisturbed regions. In this paper, we consider that the undisturbed region of the solution surface is flat for simplicity. Moreover, the initial condition of the system (1) and (2), $\mathbf{q}_0(x, 0)$, is not smooth ($C^\infty$) at $x_0$. For the traffic system, it is easy to see that the wavefront and wavefront trace can be expressed as the same equation in the $x$–$t$ plane.

To discuss traffic flow propagation stability we need to first investigate the characteristic velocity of the traffic system given by Eqs. (1) and (5). From now on, we use shortened notation to denote partial derivatives, for example, $\rho_x := \frac{\partial \rho}{\partial x}$. We also assume that the spacing function $h(\rho)$ is at least piecewise smooth. We thus rewrite Eq. (1) as

$$\rho_t + v\rho_x + \rho v_x = 0. \tag{6}$$

Consider the total differentials of $\rho(x, t)$ and $v(x, t)$ as follows:

$$\mathrm{d}\rho(x,t) = \rho_x \mathrm{d}x + \rho_t \mathrm{d}t, \tag{7}$$
$$\mathrm{d}v(x,t) = v_x \mathrm{d}x + v_t \mathrm{d}t \tag{8}$$

and write Eqs. (5)–(8) in matrix form

$$\begin{bmatrix} 1 & v & 0 & \rho \\ 0 & -\mu h'(\rho) & 1 & 0 \\ \mathrm{d}t & \mathrm{d}x & 0 & 0 \\ 0 & 0 & \mathrm{d}t & \mathrm{d}x \end{bmatrix} \begin{bmatrix} \rho_t \\ \rho_x \\ v_t \\ v_x \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{T}(v - h(\rho)) \\ \mathrm{d}\rho \\ \mathrm{d}v \end{bmatrix}.$$

Following the calculation in Whitham (1974), the characteristic velocities $v_c := \frac{\mathrm{d}x}{\mathrm{d}t}$ must satisfy

$$\det \begin{bmatrix} 1 & v & 0 & \rho \\ 0 & -\mu h'(\rho) & 1 & 0 \\ \mathrm{d}t & \mathrm{d}x & 0 & 0 \\ 0 & 0 & \mathrm{d}t & \mathrm{d}x \end{bmatrix} = 0,$$

namely,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - v\frac{\mathrm{d}x}{\mathrm{d}t} + \mu\rho h'(\rho) = 0. \tag{9}$$

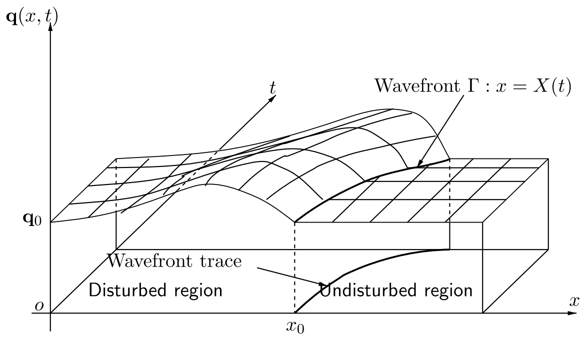


Fig. 2. A schematic of a solution surface, wavefront and wavefront trace with an initial $C^1$ discontinuous condition $\mathbf{q}(x_0, 0)$.

Therefore, the characteristic velocities $v_c$ can be found as

$$v_c = \frac{v \pm \sqrt{v^2 - 4\mu\rho h'(\rho)}}{2}. \tag{10}$$

From the above calculations we find that, if $\mu > 0$ and $h'(\rho) < 0$, one of the solutions of the characteristic velocity is larger than the average traffic flow velocity $v$. If $h'(\rho) < 0$ and we choose the downstream biasing strategy, i.e. $\Delta > 0$, then $\mu > 0$. If we consider the human driver as a special adaptive cruise control mechanism, we can compare the results found here with those found in traditional manual traffic flow models. In manual traffic systems, the case $\mu > 0$ and $h'(\rho) < 0$ is not realistic, since shock waves do not propagate to the downstream traffic in actual highways, and this fact has been remarked by Daganzo (1995) as a major deficiency of most second-order manual traffic models. From Eq. (10), the characteristic velocity $v_c \leqslant v$ iff

$$\mu \leqslant 0 \text{ and } h'(\rho) < 0, \quad \text{or} \quad h'(\rho) = 0. \tag{11}$$

The first case, $\mu \leqslant 0$ and $h'(\rho) < 0$, implies that the ACC spacing policy has an upstream or neutral biasing. If $h'(\rho) = 0$, the spacing policy is independent of highway density, which implies that, under uncongested traffic, each vehicle travels at its maximum velocity $v_f$, i.e. the traffic velocity is saturated.

## 3. Stability conditions

### 3.1. Wavefront expansion

In this section we discuss the propagation stability conditions for an ACC traffic flow model (Eqs. (1) and (5)) under large perturbations. Suppose that traffic has a constant flow density $\rho_0$ and velocity $v_0 = h(\rho_0)$. Obviously, $v_0$, $\rho_0$ are the solutions of Eqs. (1) and (5). Assume that the first derivative of the initial density profile around the wavefront (in Fig. 2) is discontinuous.[3]

Near the wavefront, we can use the time variable $t$ as a parameter and write the equation of the wavefront in the form $x = X(t)$. Using the same approach that was used to analyze manual traffic in Yi et al. (2003), it is particularly convenient to expand the solution of the system around the wavefront in powers of

$$\zeta = x - X(t), \tag{12}$$

where the wavefront has the characteristic velocity $v_c$ at the equilibrium states, i.e.

$$\dot{X}(t) = v_c(\rho_0, v_0) = v_0 + u_0 = v_0 + \frac{-v_0 \pm \sqrt{v_0^2 - 4\mu\rho_0 h'(\rho_0)}}{2}, \tag{13}$$

where $u_0 := \frac{-v_0 \pm \sqrt{v_0^2 - 4\mu\rho_0 h'(\rho_0)}}{2}$.

Using Eq. (12), we can expand the flow variables $\rho$ and $v$ behind the wavefront in a power series of $\zeta$ as

$$\rho(x,t) = \rho_0 + \zeta\rho_1(t) + \frac{1}{2}\zeta^2\rho_2(t) + \cdots, \tag{14}$$

$$v(x,t) = v_0 + \zeta v_1(t) + \frac{1}{2}\zeta^2 v_2(t) + \cdots, \tag{15}$$

where

$$\rho_i(t) = \left.\frac{\partial^i \rho}{\partial x^i}\right|_{(X(t)^-,t)}, \quad v_i(t) = \left.\frac{\partial^i v}{\partial x^i}\right|_{(X(t)^-,t)}, \qquad i = 1, 2, 3, \ldots$$

[3] This assumption can be generalized. For example, if the $m$th derivatives of $\rho$ and $v$ are the first ones to be discontinuous, the expanded power series (Eqs. (14) and (15)) beyond $\rho_0$ and $v_0$ start with the term in $\zeta^m$.

We now calculate the partial derivatives of state variables $\rho$ and $v$, using Eqs. (14) and (15),

$$\rho_t = -\dot{X}(t)\rho_1(t) + \zeta\dot{\rho}_1(t) + \zeta[-\dot{X}(t)]\rho_2(t) + \frac{1}{2}\zeta^2\dot{\rho}_2(t) + \cdots, \tag{16a}$$

$$\rho_x = \rho_1(t) + \zeta\rho_2(t) + \frac{1}{2}\zeta^2\rho_3(t) + \cdots, \tag{16b}$$

$$v_t = -\dot{X}(t)v_1(t) + \zeta\dot{v}_1(t) + \zeta[-\dot{X}(t)]v_2(t) + \frac{1}{2}\zeta^2\dot{v}_2(t) + \cdots, \tag{16c}$$

$$v_x = v_1(t) + \zeta v_2(t) + \frac{1}{2}\zeta^2 v_3(t) + \cdots \tag{16d}$$

Similarly, for $h(\rho(x,t))$ and $h'(\rho(x,t))$, we obtain

$$h(\rho) = h(\rho_0) + \zeta h'(\rho_0)\rho_1 + \frac{1}{2}\zeta^2 h''(\rho_0) + \cdots, \tag{17a}$$

$$h'(\rho) = h'(\rho_0) + \zeta h''(\rho_0)\rho_1 + \frac{1}{2}\zeta^2 h'''(\rho_0) + \cdots. \tag{17b}$$

Substituting Eqs. (16) and (13) into Eq. (6) and arranging terms into successive powers of $\zeta$ gives[4]

$$[-\rho_1(v_0+u_0) + \rho_1 v_0 + \rho_0 v_1] + \zeta[\dot{\rho}_1 - \rho_2(v_0+u_0) + 2\rho_1 v_1 + \rho_2 v_0 + \rho_0 v_2] + \cdots = 0.$$

Thus, for the coefficients of the first two terms $\zeta^0$ and $\zeta^1$, we obtain

$$-\rho_1 u_0 + \rho_0 v_1 = 0, \tag{18a}$$

$$\dot{\rho}_1 - \rho_2 u_0 + 2\rho_1 v_1 + \rho_0 v_2 = 0. \tag{18b}$$

Similarly, substituting Eqs. (16) and (17) into Eq. (5), we obtain

$$-(v_0+u_0)v_1 - \mu h'(\rho_0)\rho_1 = 0, \tag{19a}$$

$$\dot{\rho}_1 - v_2(v_0+u_0) - \mu\left[\rho_1^2 h''(\rho_0) + \rho_2 h'(\rho_0)\right] + \frac{1}{T}[v_1 - \rho_1 h'(\rho_0)] = 0. \tag{19b}$$

From Eqs. (18a) and (19a) we can eliminate $\rho_1$ and $v_1$

$$u_0^2 + u_0 v_0 + \mu h'(\rho_0)\rho_0 = 0. \tag{20}$$

Thus,

$$u_0 = \frac{-v_0 \pm \sqrt{v_0^2 - 4\mu\rho_0 h'(\rho_0)}}{2},$$

which is the same as the expected result in Eq. (13).

For Eqs. (18b) and (19b) notice that the coefficients of terms $\rho_2$ and $v_2$ are linearly dependent since

$$\det\begin{bmatrix} -u_0 & \rho_0 \\ -\mu h'(\rho_0) & -(u_0+v_0) \end{bmatrix} = u_0^2 + u_0 v_0 + \mu h'(\rho_0)\rho_0 = 0.$$

Therefore, we can eliminate $\rho_2$ and $v_2$, plug $\rho_1 = \frac{\rho_0}{u_0}v_1$ in Eqs. (18b) and (19b), and then obtain

$$\dot{v}_1 + \alpha v_1 + \beta v_1^2 = 0, \tag{21}$$

where

$$\alpha = \frac{1}{T}\frac{\rho_0 u_0 h'(\rho_0) - u_0^2}{\rho_0\mu h'(\rho_0) - u_0^2} = \frac{1}{T}\frac{u_0 - \rho_0 h'(\rho_0)}{2u_0 + v_0}, \qquad \beta = \frac{\rho_0\mu[2h'(\rho_0) + \rho_0 h''(\rho_0)]}{\rho_0\mu h'(\rho_0) - u_0^2}. \tag{22}$$

For the dynamic system (21) we have

$$\dot{v}_1 = -v_1(\alpha + \beta v_1).$$

[4] For simplicity we neglect time dependence in the variable notation.

Notice that $v_1(t) = \frac{\partial v(x,t)}{\partial x}\Big|_{(X(t)^-,t)}$, namely, the slope of points along the wavefront trace. The above equation gives the slope evolution at the wavefront. The propagation stability of Eq. (21) can thus be analyzed in terms of the initial condition $v_1(0)$ and the parameters $\alpha$ and $\beta$. Table 1 shows the stability conditions of the system given by Eq. (21).

## 3.2. Stability conditions

In this section, we discuss the stability conditions for different choices of the parameter $\mu$, i.e. relaxation time $T$ and biasing distance $\Delta$, and the ACC spacing policy $h(\rho)$.

(1) $T > 0$, $\Delta > 0$.
In this case, $\mu = \frac{\Delta}{T} > 0$, and we have

- If $h'(\rho_0) = 0$, i.e. under uncongested traffic flow and assuming that each vehicle travels at its maximum velocity $v_f$, we obtain

$$\alpha = \frac{1}{T} > 0, \qquad \beta = 0.$$

From Table 1, we can conclude that the system is asymptotically stable for any initial condition of the perturbations.

- If $h'(\rho_0) < 0$, we can choose

$$u_0 = \frac{-v_0 - \sqrt{v_0^2 - 4\rho_0\mu h'(\rho_0)}}{2} < 0,$$

because of $v_c = v_0 + u_0 < v_0$ and the fact that it is realistic to keep the perturbation wave propagating upstream along the traffic flow. Let $\delta := \sqrt{v_0^2 - 4\rho_0\mu h'(\rho_0)} > 0$ and $u_0 = -\frac{v_0+\delta}{2}$. Thus, we obtain

$$\rho_0\mu h'(\rho_0) - u_0^2 = u_0\delta$$

and

$$\alpha = \frac{\rho_0 h'(\rho_0) - u_0}{T\delta}, \qquad \beta = \frac{\rho_0\mu[2h'(\rho_0) + \rho_0 h''(\rho_0)]}{u_0\delta}. \tag{23}$$

From Table 1, in order to achieve asymptotic stability for any initial condition $v_1(0) \geqslant -\frac{\alpha}{\beta}$, we need (i) $\beta > 0$, $\alpha > 0$, or (ii) $\beta > 0$, $\alpha = 0$, or (iii) $\beta = 0$, $\alpha > 0$. Before we discuss each case in detail, it is interesting to calculate the conditions on $h(\rho)$ to attain stability when $\alpha = 0$ or $\beta = 0$.

If $\alpha = 0$, we obtain

$$\rho_0 h'(\rho_0) - u_0 = \rho_0 h'(\rho_0) + \frac{v_0 + \sqrt{v_0^2 - 4\rho_0\mu h'(\rho_0)}}{2} = 0 \tag{24}$$

Table 1
Stability conditions for dynamical system Eq. (21)

| Parameters $\alpha$ and $\beta$ | Stable region | Unstable region |
|---|---|---|
| $\beta > 0$, $\alpha > 0$ | $v_1(0) \in \left[-\frac{\alpha}{\beta}, \infty\right)$, $v^1(t) \to 0$ | $v_1(0) \in \left(-\infty, -\frac{\alpha}{\beta}\right)$, $v^1(t) \to -\infty$ |
| $\beta > 0$, $\alpha = 0$ | $v_1(0) \in \mathbb{R}$, $v_1(t) \to 0$ | $\varnothing$ |
| $\beta > 0$, $\alpha < 0$ | $v_1(0) \in [0, \infty)$, $v_1(t) \to -\frac{\alpha}{\beta}$ | $v_1(0) \in (-\infty, 0)$, $v_1(t) \to -\infty$ |
| $\beta < 0$, $\alpha > 0$ | $v_1(0) \in (-\infty, -\frac{\alpha}{\beta})$, $v_1(t) \to 0$ | $v_1(0) \in (-\frac{\alpha}{\beta}, \infty)$, $v_1(t) \to \infty$ |
| $\beta < 0$, $\alpha = 0$ | $\varnothing$ | $v_1(0) \in \mathbb{R}$, $v_1(t) \to \infty$ |
| $\beta < 0$, $\alpha < 0$ | $v_1(0) \in (-\infty, 0)$, $v_1(t) \to -\frac{\alpha}{\beta}$ | $v_1(0) \in (0, \infty)$, $v_1(t) \to \infty$ |
| $\beta = 0$, $\alpha > 0$ | $v_1(0) \in \mathbb{R}$, $v_1(t) \to 0$ | $\varnothing$ |
| $\beta = 0$, $\alpha = 0$ | $v_1(0) \in \mathbb{R}$, $v_1(t) \equiv v_1(0)$ | $\varnothing$ |
| $\beta = 0$, $\alpha < 0$ | $\varnothing$ | $v_1(0) \in \mathbb{R}$, $v_1(t) \to \infty$ |

by noticing that $v_0 = h(\rho_0)$, Eq. (24) can be arranged as a first-order ordinary differential equation (ODE) of the function $h(\rho)$

$$\rho h'(\rho) + h(\rho) + \mu = 0.$$

Solving the above equation with the physical constrained that $h\left(\frac{1}{L_v}\right) = 0$, we obtain

$$h(\rho) = \frac{\mu}{L_v \rho} - \mu = \frac{\mu}{L_v}\left(\frac{1}{\rho} - L_v\right), \tag{25}$$

which is exactly the CTH policy given by Eq. (4) with a constant headway time $h_w = \frac{L_v}{\mu}$! Moreover, if we apply the same calculation when $\beta = 0$, we can find that the spacing policy function $h(\rho)$ must satisfy

$$h(\rho) = \frac{c_1}{\rho} + c_2, \tag{26}$$

where $c_1$ and $c_2$ are constants determined by the constrained conditions imposed on $h(\rho)$. Not surprisingly, the function form given by Eq. (26) is same as Eq. (4). Thus, we can conclude that, if spacing policy $h(\rho)$ satisfies the conditions that $\alpha = 0$, then it must also satisfy $\beta = 0$. However, the converse is not always true because it depends on whether $h_w = \frac{L_v}{\mu}$.

The above analysis rules out case (ii) because, if $\alpha = \beta = 0$, the system is marginally stable but not asymptotically stable. We need to discuss the other two cases separately. For case (i), $\beta > 0$, $\alpha > 0$, from Eqs. (23) and (24), we need

$$\begin{cases} \rho_0 h'(\rho_0) + h(\rho_0) + \mu > 0, \\ 2h'(\rho_0) + \rho_0 h''(\rho_0) < 0. \end{cases}$$

Notice that $v_0 = h(\rho_0)$ and $q_0 = \rho_0 v_0 = \rho_0 h(\rho_0)$. Then we can rewrite the above inequalities as

$$\begin{cases} \left.\frac{dq}{d\rho}\right|_{\rho_0} + \mu > 0, \\ \left.\frac{d^2 q}{d\rho^2}\right|_{\rho_0} < 0. \end{cases} \tag{27}$$

The second condition given in (27) reminds us of the concave condition for the flow–density ($q$–$\rho$) relationship in manual traffic flow (del Castillo et al., 1994). In this case, when we design the ACC policy, we still need to satisfy this condition for ACC traffic flow. The first condition in (27) implies that the slope of the $q$–$\rho$ curve must be greater than $-\mu$.

For case (iii), $\beta = 0$, $\alpha > 0$, it is easy to see that the ACC policy must be the CTH given by Eq. (26) and, under such a CTH policy, the constant time headway constant $h_w$ must be chosen such that

$$\frac{L_v}{h_w} < \mu, \tag{28}$$

in order to guarantee the condition $\alpha > 0$ as

$$\rho h'(\rho) + h(\rho) + \mu > 0. \tag{29}$$

**Summary:** In the case when $T > 0$, $\Delta > 0$, i.e. $\mu > 0$, the ACC traffic flow is asymptotically stable for an equilibrium state iff

(i) traffic flow travels at velocity $v_f$;
(ii) the CTH policy is used with that the constant headway time $h_w$ is chosen such that $\frac{L_v}{h_w} < \mu$;
(iii) for other ACC spacing policies, if the proposed flow–density relationship, $q$–$\rho$ curve, is concave and its slope is larger than $-\mu$ at this equilibrium state.

(2) $T > 0$, $\Delta = 0$.

In this case $\mu = 0$, and thus $\beta = 0$. In order to guarantee asymptotic stability for any initial condition of $v_1(0)$, from Table 1 we need $\alpha > 0$. From Eq. (9), we have $u_0 = 0$ or $u_0 = -v_0$. However, note that $u_0 = 0$ is impossible because, if $u_0 = 0$, from Eq. (18a) we have that $v_1(t) = 0$, which is not true for the perturbed velocity profile. Thus, $u_0 = -v_0$, and from (22) we have

$$\alpha = \frac{1}{T}\frac{\rho_0 u_0 h'(\rho_0) - u_0^2}{-u_0^2} = \frac{1}{T}\frac{v_0 + \rho_0 h'(\rho_0)}{v_0} = \frac{1}{Tv_0}\left.\frac{\mathrm{d}q}{\mathrm{d}\rho}\right|_{\rho_0}.$$

The stability condition is then

$$h'(\rho_0) = 0, \quad \text{or} \quad \left.\frac{\mathrm{d}q}{\mathrm{d}\rho}\right|_{\rho_0} > 0. \tag{30}$$

**Summary**: In the case when $T > 0$, $\Delta = 0$, i.e. $\mu = 0$, the ACC traffic flow is asymptotically stable for an equilibrium state iff

(i) traffic flow travels at velocity $v_f$;
(ii) under the ACC spacing policy, the slope of the proposed flow–density relationship, $q$–$\rho$ curve, is positive at this equilibrium state.

(3) $T > 0$, $\Delta < 0$.
In this case $\mu < 0$, similarly to the first case when $\mu > 0$, if $h'(\rho_0) = 0$, i.e. an uncongested traffic flow condition, the system is asymptotically stable; if $h'(\rho_0) < 0$. Then

$$u_0 = \frac{-v_0 \pm \sqrt{v_0^2 - 4\rho_0\mu h'(\rho_0)}}{2} = \frac{-v_0 \pm \delta}{2} < 0,$$

where $\delta = \sqrt{v_0^2 - 4\rho_0\mu h'(\rho_0)}$. We can follow the same analysis as in the first case of $\mu > 0$, except that, in this case, the values of $u_0$ can take two values.

For the case when $\beta > 0$, $\alpha > 0$, if $u_0 = \frac{-v_0-\delta}{2}$, from Eq. (23), we obtain the following stability conditions:

$$\begin{cases} \left.\frac{\mathrm{d}q}{\mathrm{d}\rho}\right|_{\rho_0} + \mu > 0, \\ \left.\frac{\mathrm{d}^2q}{\mathrm{d}\rho^2}\right|_{\rho_0} > 0. \end{cases} \tag{31}$$

If $u_0 = \frac{-v_0+\delta}{2}$, using Eq. (22) we can similarly obtain

$$\begin{cases} \left.\frac{\mathrm{d}q}{\mathrm{d}\rho}\right|_{\rho_0} + \mu > 0, \\ \left.\frac{\mathrm{d}^2q}{\mathrm{d}\rho^2}\right|_{\rho_0} < 0. \end{cases} \tag{32}$$

Combining (31) and (32), we have

$$\begin{cases} \left.\frac{\mathrm{d}q}{\mathrm{d}\rho}\right|_{\rho_0} + \mu > 0, \\ \left.\frac{\mathrm{d}^2q}{\mathrm{d}\rho^2}\right|_{\rho_0} \neq 0. \end{cases} \tag{33}$$

For the case when $\beta = 0$, $\alpha > 0$, this implies the CTH policy and both choices of $u_0$ give the same condition:

$$-\frac{L_\mathrm{v}}{h_\mathrm{w}} + \mu > 0$$

which is never satisfied for $\mu < 0$!

**Summary:** In the case when $T > 0$, $\Delta < 0$, i.e. $\mu < 0$, the ACC traffic flow is asymptotically stable for an equilibrium state iff,

(i) traffic flow travels at velocity $v_f$;
(ii) under the ACC spacing policy, if the proposed flow–density relationship, $q$–$\rho$ curve, is either concave or convex and its slope is larger than $-\mu$ at this equilibrium state.

Table 2
Stability conditions under different model parameters

| | $T > 0$ | | | $T = 0$ | | |
|---|---|---|---|---|---|---|
| | $\Delta > 0$ ($\mu > 0$) | $\Delta = 0$ ($\mu = 0$) | $\Delta < 0$ ($\mu < 0$) | $\Delta > 0$ | $\Delta = 0$ | $\Delta < 0$ |
| Asymp. stab. conds. | $0 \leqslant \rho_0 \leqslant \rho_{\min}$; $\left.\frac{d^2q}{d\rho^2}\right\|_{\rho_0} \leqslant 0$ and $\left.\frac{dq}{d\rho}\right\|_{\rho_0} + \mu > 0$ | $0 \leqslant \rho_0 \leqslant \rho_{\min}$; $\left.\frac{dq}{d\rho}\right\|_{\rho_0} > 0$ | $0 \leqslant \rho_0 \leqslant \rho_{\min}$; $\left.\frac{d^2q}{d\rho^2}\right\|_{\rho_0} \neq 0$ and $\left.\frac{dq}{d\rho}\right\|_{\rho_0} > \|\mu\|$ | $0 \leqslant \rho_0 \leqslant \rho_{\min}$; $\left.\frac{d^2q}{d\rho^2}\right\|_{\rho_0} \leqslant 0$; | $0 \leqslant \rho_0 \leqslant \rho_{\min}$ | $0 \leqslant \rho_0 \leqslant \rho_{\min}$ |
| CTH policy | Asym. stab. iff $\frac{L_v}{h_w} < \mu$ | Always unstable | Always unstable | Asymptotically stable | Marginally stable | Always unstable |

(4) $T = 0$ and $\Delta \neq 0$.
In this case $\mu = \infty$. We can apply the results given in the previous cases by letting $T \to 0$ for each case of $\Delta$.

- $\Delta > 0$, $T \to 0$ implies $\mu \to +\infty$. From case (1) discussed above, it is easy to obtain that: (1) the ACC traffic flow is asymptotically stable for the CTH policy; (2) for any ACC spacing policy, if the $q$–$\rho$ relationship is concave at an equilibrium state, then the state is asymptotically stable; (3) any uncongested traffic equilibrium state, i.e. traffic flow travels at velocity $v_f$, is asymptotically stable.
- $\Delta < 0$, $T \to 0$ implies $\mu \to -\infty$. From case (3) discussed above, it is easy to conclude that the system is asymptotically stable only if for an uncongested traffic equilibrium state, i.e. traffic flow travels at velocity $v_f$.

(5) $T = 0$ and $\Delta = 0$.
In this case, the system (1) and (5) reduces to a first-order nonlinear PDE system, similar to the LWR model (Lighthill and Whitham, 1955), which is stable in the sense of Lyapunov, or marginally stable. Any disturbance will propagate along the wavefront without any amplifications or attenuation (Whitham, 1974).

We have discussed the stability conditions for system (1) and (5) under all possible combinations of model parameters $\Delta$ and $T \geqslant 0$. The results can be summarized as Table 2.

The stability analysis and conditions in Table 2 give us quantitatively traffic flow or velocity conditions under which the system given by Eqs. (1) and (5) is asymptotically propagation stable. They also describe how the model parameters affect traffic flow systems. However, it is not trivial to apply such conditions to a real ACC traffic flow system because we need the knowledge of the exact model structure and its parameters, such as relaxation time $T$ and biasing strategy distance $\Delta$.

## 4. Comparisons and discussions

In this section, we compare the results presented in the previous section with those reported in Swaroop and Rajagopal (1999), Li and Shrivastava (2002) and Wang and Rajamani (2004). Here we only consider a dense traffic equilibrium density $\rho_0 > \rho_{\min}$.[5]

In Swaroop and Rajagopal (1999), the authors considered the ACC system dynamics on the traffic flow by a first-order regulator with an exponential decaying rate $\frac{1}{\tau}$. The traffic flow dynamics are given in PDE form as

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \\ \frac{\partial v}{\partial t} + (v + \rho h'(\rho)) \frac{\partial v}{\partial x} = -\frac{1}{\tau}(v - h(\rho)). \end{cases} \tag{34}$$

Comparing these dynamics with the traffic flow dynamics (1) and (5) discussed in this paper, the first equations are the same. One difference between the second equations, i.e. the velocity dynamics, comes from the fact that we treat the spatial biasing strategy by introducing $\Delta$ for the ACC policy, while in Swaroop and Rajagopal (1999) no such consideration exists for a spatially continuous model. The use of spatial biasing strategy $\Delta$

[5] When a traffic equilibrium density $\rho_0 \leqslant \rho_{\min}$, it is always asymptotically stable for any ACC spacing policies.

allows us to consider general ACC spacing policies. Thus, the velocity dynamics in (34) has no density gradient term $\frac{\partial\rho}{\partial x}$, which does appear in (5). Another difference comes from the method used to abstract the ACC system dynamics. In Swaroop and Rajagopal (1999), the authors took the total derivative of traffic flow velocity, $\dot{v}(x,t) = v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}$, as the controlled *vehicle* acceleration, using a Lagrangian coordinate system that is fixed with the moving vehicle. In this paper we consider the partial derivative of traffic flow with respect to time, $\frac{\partial v(x,t)}{\partial t}$, as the controlled acceleration of the *traffic flow* at a fixed position $x$, using the Euler coordinate system. Similar Euler coordinate-based micro-mesoscopic models were derived by Liu et al. (1998) and Zhang (1998) for manual traffic flow models.

Even though there exists minor difference in the system dynamics, the stability conditions derived in this paper are consistent with those found in Swaroop and Rajagopal (1999). In Swaroop and Rajagopal (1999), a linearized stability analysis was discussed for the PDE system and the authors only concluded that, when $\tau \to 0$, system (34) behaves like the LWR model for manual traffic systems, which is marginally stable (i.e. stable in the sense of Lyapunov). Moreover, the authors also found that "*small density disturbances propagate upstream without any attenuation*". If we consider $\Delta = 0$ and let $T \to 0$, from Table 2, we obtain the same conclusion.

We can further investigate the stability conditions for the systems given by (34). We can consider a more general velocity dynamics of the traffic flow of ACC vehicles by modifying the dynamics (2) for dense flow as follows:

$$v(x(t+T), t+T) = h(\rho(x(t+T)+\Delta, t+T)). \tag{35}$$

In the above velocity dynamics, we consider that the vehicle velocity of ACC traffic flow at position $x(t+T)$ and time $t+T$ is regulated by the density at position $x(t+T)+\Delta$ and time $t+T$. The difference between the dynamics (2) and (35) is that we take the position variable $x$ to be changing with time $t$ in the microscopic level (individual vehicle) instead of traffic flow velocity at a fixed position $x$ in the macroscopic level (as in Eq. (2)). We can expand Eq. (35) using Taylor series and obtain

$$v(x,t) + T\frac{\partial v}{\partial t} + Tv\frac{\partial v}{\partial x} = h(\rho) + h'vT\frac{\partial\rho}{\partial x} + h'\Delta\frac{\partial\rho}{\partial x} + h'T\frac{\partial\rho}{\partial t}, \tag{36}$$

where we have neglected higher-order terms of partial derivatives. Using the conservation law (1), $\frac{\partial\rho}{\partial t} = -v\frac{\partial\rho}{\partial x} - \rho\frac{\partial v}{\partial x}$, Eq. (36) becomes

$$\frac{\partial v}{\partial t} + (v + \rho h')\frac{\partial v}{\partial x} - \mu h'\frac{\partial\rho}{\partial x} = -\frac{1}{T}(v - h(\rho)), \tag{37}$$

where $\mu = \frac{\Delta}{T}$. It is easy to see that when $\Delta = 0$, i.e. $\mu = 0$, Eq. (37) reduces to Eq. (34).

For the hyperbolic system (1) and (37), we can apply the same process of the wavefront expansion technique to check the stability conditions. First, the system characteristics can be calculated as

$$v_c := \frac{dx}{dt} = v + \frac{1}{2}\left[\rho h' \pm \sqrt{(\rho h')^2 - 4\mu\rho h'}\right] = v + u, \tag{38}$$

where $u = \frac{1}{2}[\rho h' \pm \sqrt{(\rho h')^2 - 4\mu\rho h'}]$. The Riccati equation is the same as Eq. (21) with

$$\alpha = \frac{u_0^2\frac{h''(\rho_0)}{h'(\rho_0)} - \frac{1}{T}\mu}{u_0(u_0 - 2\mu)}, \qquad \beta = \frac{2u_0 - 3\mu - \rho_0\mu\frac{h''(\rho_0)}{h'(\rho_0)}}{u_0 - 2\mu}, \tag{39}$$

where $u_0 = u(\rho_0)$.

If we consider the case of the ACC CTH policy given by Swaroop and Rajagopal (1999), then $\mu = 0$, $h(\rho) = \frac{1}{h_w}(\frac{1}{\rho} - L_v)$, and we obtain

$$\alpha = -\frac{2}{\rho_0} < 0, \qquad \beta = 2 > 0. \tag{40}$$

From Table 1, we can conclude that system is marginal stable and moreover, the slope of the disturbance along the wavefront converges to a constant $-\frac{\alpha}{\beta} = \frac{1}{\rho_0}$, which is consistent with the result obtained by Swaroop and Rajagopal (1999).

In Li and Shrivastava (2002), the authors discussed the stability of a circular traffic flow under an ACC CTH policy (4). A spatially biasing strategy was used to model the ACC spacing policy. The spatially continuous model in Li and Shrivastava (2002) is given by

$$\begin{cases} \frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0, \\ v(x,t)=\frac{1}{h_{\mathrm{w}}}\left(\frac{1}{\rho(x+\Delta(x,t),t)}-L_{\mathrm{v}}\right), \end{cases} \tag{41}$$

where $\Delta(x,t)$ could be positive, zero or negative, depending on whether the spatially biasing strategy is downstream, neutral and upstream, respectively. The velocity dynamics in (41) do not consider the ACC system dynamics and the author assumed that "*an infinite vehicle speed can be achieved*". A coordinate transformation (from an Euler framework to a Lagrangian framework) and a Lyapunov function was used to prove the $L_2$-stability for a circular highway. In Li and Shrivastava (2002), the stability conclusions for various biasing strategies are: for downstream biasing ($\Delta>0$), system is asymptotically stable; for neutral biasing ($\Delta=0$), system is marginally stable; and for upstream biasing ($\Delta<0$), system is unstable.

Note that, if we let $T=0$ for the model (1) and (5) used in this paper, we obtain exactly the same model (41) given in Li and Shrivastava (2002). Therefore, applying $T=0$ for the stability results given in Table 2 and the ACC CTH policy (4), we reach the same stability conclusion.

In Wang and Rajamani (2004), the authors did not use a spatially continuous PDE model to investigate the stability. Instead, a spatially biasing discrete model was used. The authors considered the ACC system dynamics as a first-order regulator with decaying time $\frac{1}{\lambda}$. The authors concluded that, when $\frac{\mathrm{d}q}{\mathrm{d}\rho}>0$ at the equilibrium state, then the traffic flow was stable for all of boundary conditions, or *so-called unconditional traffic flow stability*. The authors applied this stability criterion to the neutral biasing ACC CTH policy and a conclusion of instability was reached because of $\frac{\mathrm{d}q}{\mathrm{d}\rho}=-\frac{L_{\mathrm{v}}}{h_{\mathrm{w}}}<0$. Moreover, the authors suggested an alternative variable time-gap (VTG) spacing policy under which the traffic flow was unconditionally stable within a critical density value. The VTG policy proposed in Wang and Rajamani (2004) is given by

$$v=v_{\mathrm{f}}\left(1-\frac{\rho}{\rho_{\max}}\right), \tag{42}$$

where $v_{\mathrm{f}}$ is the free velocity and $\rho_{\max}$ is the maximum density. Note that the VTG given by Eq. (42) is same as the fundamental diagram in manual traffic flow.

In Wang and Rajamani (2004), the authors used the neutral biasing strategy to check the stability of the ACC CTH policy. The stability condition, $\frac{\mathrm{d}q}{\mathrm{d}\rho}>0$, is consistent with the results we obtained under $T>0$ and $\Delta=0$ in Table 2. However, when Wang and Rajamani (2004) proved the stability condition $\frac{\mathrm{d}q}{\mathrm{d}\rho}>0$, an upstream biasing strategy, $0.5<\alpha_i\leqslant 1$, was used in the spatially discrete model.[6] In this case, the stability condition resulting from our analysis is $\left.\frac{\mathrm{d}q}{\mathrm{d}\rho}\right|_{\rho_0}>|\mu|$, which is different from the condition $\frac{\mathrm{d}q}{\mathrm{d}\rho}>0$ given by Wang and Rajamani (2004). In order to investigate this discrepancy, we take the proposed VTG policy in Wang and Rajamani (2004) as an example, and apply the two stability criteria. Fig. 3(a) shows the flow–density relationship ($q$–$\rho$ curve) resulting from Eq. (42). Notice that

$$q=\rho v=v_{\mathrm{f}}\left(\rho-\frac{\rho^2}{\rho_{\max}}\right) \Rightarrow \frac{\mathrm{d}q}{\mathrm{d}\rho}=v_{\mathrm{f}}\left(1-\frac{2\rho}{\rho_{\max}}\right).$$

The stability condition given in Wang and Rajamani (2004) tells us that, if the traffic density $\rho_0\in[0,\frac{\rho_{\max}}{2}]$, then $\rho_0$ is asymptotically stable; while the stability condition in Table 2 states that if the traffic density $\rho_0\in[0,\rho_{\mathrm{c}}]$, then $\rho_0$ is asymptotically stable, where $\rho_{\mathrm{c}}=\frac{\rho_{\max}}{2}\left(1-\frac{|\mu|}{v_{\mathrm{f}}}\right)<\frac{\rho_{\max}}{2}$ for $\mu\neq 0$. Notice that at the density $\frac{\rho_{\max}}{2}$, the traffic system has its maximum flow rate $q_{\mathrm{m}}$, and the stability condition $\frac{\mathrm{d}q}{\mathrm{d}\rho}>0$ tells that, before the traffic flow rate reaches the maximum value $q_{\mathrm{m}}$, it is asymptotically stable (Fig. 3(a)). This is not a case from observations of the real traffic flow data. Fig. 3(b) shows the flow–density relationship at one spot on the Southern

[6] The biasing strategy in the spatially discrete model is determined by magnitude of the weighting coefficient $\alpha_i$, $0\leqslant\alpha_i\leqslant 1$, which is used in calculating the traffic flow rate $q_i$ crossing the $i$th section, $q_i=\alpha_i\rho_i v_i+(1-\alpha_i)\rho_{i+1}v_{i+1}$. If $0\leqslant\alpha_i<0.5$, downstream biasing; $\alpha_i=0.5$, neutral biasing; $0.5<\alpha_i\leqslant 1$, upstream biasing. Same spatial biasing strategy rule was utilized in Swaroop and Rajagopal (1999), Li and Shrivastava (2002) and Wang and Rajamani (2004).

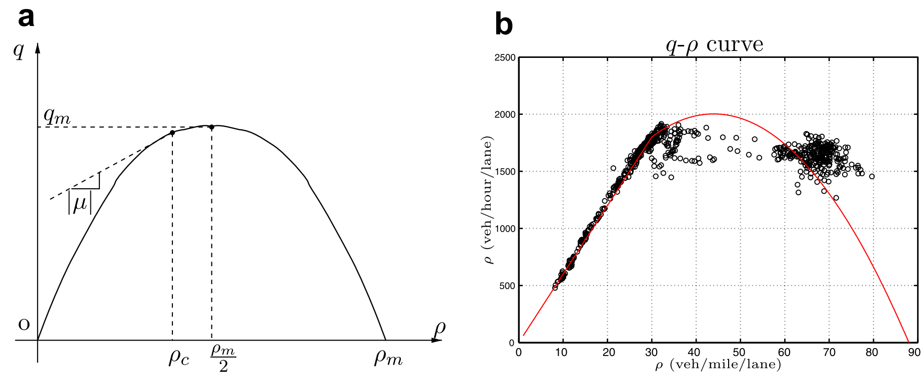


Fig. 3. Flow-density relationship ($q$–$\rho$ curve): (a) VTG policy as Eq. (42); (b) traffic data of highway I-210 (dot) and fitting curve (solid).

California I-210 on April 25, 2001. By curve fitting Eq. (42), we obtain that $\rho_{\max} = 88$ (veh/mile/lane) and $v_f = 60$ mph. From the data, we can roughly see that before the traffic flow reaches its maximum volume $q_m$ ($\approx$2200 veh/hour/lane for highway I-210) and critical density $\rho_{\max}/2$ ($\approx$44 veh/mile/lane), the traffic density becomes unstable around a critical density $\rho_c$ ($\approx$33 veh/mile/lane) and jumps to the unstable branch. The data in Fig. 3 is from a manual traffic flow; however, since the VTG policy (42) is also used to model the manual traffic velocity–density relationship, we believe that a similar behavior will be observed for ACC vehicles controlled under the spacing policy (42).

If we consider the stability condition $\frac{dq}{d\rho} > 0$ in Wang and Rajamani (2004) for downstream biasing strategies, we can find that it is a sufficient condition. In this case, $T > 0$ and $\Delta > 0$, from Table 2, we can find that the conditions for traffic asymptotic stability are $\frac{dq}{d\rho} + \mu > 0$ and $\frac{d^2q}{d\rho^2} \leqslant 0$. The latter condition is a fundamental requirement for the $q$–$\rho$ relationship in manual traffic flow (Ansorge, 1990; del Castillo et al., 1994). For the first condition, it is automatically satisfied if $\frac{dq}{d\rho} > 0$ with the condition $\mu = \frac{\Delta}{T} > 0$.

## 5. Conclusion

In this paper we discussed the problem whether a traffic flow perturbation will propagate and form a traffic shockwave for adaptive cruise controlled (ACC) vehicles. A concept of traffic flow propagation stability is proposed and compared with the existing traffic flow stability results. A generalized macroscopic traffic model with velocity saturation for the ACC traffic flow system was proposed. A generalized stability criterion for a asymptotically stable traffic flow system was derived using a wavefront expansion method. We discussed and compared this stability criterion with those previously obtained by other authors. We found that the stability results derived in this paper covered all stability conditions obtained for ACC spacing policies in addition to the constant time headway (CTH) policy. Moreover, the nonlinear stability analysis in this paper provided more precise stability information under perturbations than the previous approximate linearized stability approach. The stability condition derived in this paper can be used for prediction of ACC traffic flow stability, and to perform a quantitative stability analysis for the ACC spacing policies.

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## References

Ansorge, R., 1990. What does the entropy condition mean in traffic flow theory? Transportation Research B 24 (2), 133–143.
Daganzo, C., 1995. Requiem for second-order fluid approximations of traffic flow. Transportation Research B 29 (4), 277–286.

del Castillo, J.M., 2001. Propagation of perturbations in dense traffic flow: a model and its implications. Transportation Research B 35 (4), 367–389.

del Castillo, J.M., Pintado, P., Benítez, F.G., 1994. A formulation for the reaction time of traffic flow models. Transportation Research B 28 (1), 35–60.

Holland, E., 1998. A generalized stability criterion for motorway traffic. Transportation Research B 32 (2), 141–154.

Li, P., Shrivastava, A., 2002. Traffic flow stability induced by constant time headway policy for adaptive cruise control vehicles. Transportation Research C 10 (4), 275–301.

Lighthill, M.J., Whitham, J.B., 1955. On kinematic waves II: a theory of traffic flow on long crowded roads. Proceedings of the Royal Society A 229, 317–345.

Liu, G., Lyrintzis, A., Michalopoulos, P., 1998. Improved high-order model for freeway traffic flow. Transportation Research Board 1644, 37–46.

Swaroop, D., Rajagopal, K.R., 1999. Intelligent cruise control systems and traffic flow stability. Transportation Research C 7 (6), 329–352.

Wang, J., Rajamani, R., 2004. Should adaptive cruise control (ACC) systems be designed to maintain a constant time-gap between vehicles? IEEE Transactions on Vehicular Technology 53 (5), 1480–1490.

Whitham, G.B., 1974. Linear and Nonlinear Waves. John Wiley and Sons Inc., New York, NY.

Yi, J., Lin, H., Alvarez, L., Horowitz, R., 2003. Macroscopic traffic flow stability through wavefront expansion. Transportation Research B 37 (7), 661–679.

Zhang, H.M., 1998. A theory of nonequilibrium traffic flow. Transportation Research B 32 (7), 485–498.

Zhang, H.M., 1999. Analyses of the stability and wave properties of a new continuum traffic theory. Transportation Research B 33 (6), 399–415.