On stable simultaneous input and state estimation for discrete-time linear systems

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SUMMARY

This work is devoted to solving simultaneous input and state estimation (SISE) problem for discrete-time linear systems. Our aim is to develop stable SISE algorithms. By applying the minimum variance unbiased estimation technique, we derive two SISE algorithms in the presence or absence of direct feedthrough, respectively. Riccati-like equations are formulated and presented to analyze the stability conditions of the proposed algorithms. Simulation examples are provided to further illustrate the effectiveness of the proposed algorithms and support the theoretical findings. Copyright © 2011 John Wiley & Sons, Ltd.

Received 15 October 2009; Revised 8 December 2010; Accepted 10 December 2010

KEY WORDS: unknown input estimation; state estimation; unbiased estimation; minimum variance; convergence property

1. INTRODUCTION

The problem of simultaneous input and state estimation (SISE) for dynamic systems has a wide range of applications, such as fault detection and diagnosis [1], maneuvering target tracking [2], geophysics and environmentology [3], etc. In these applications, inputs and state variables are often unmeasurable or inaccessible. SISE can also find applications in networked control systems with unknown input package delays and even losses [4]. Because of its practical significance, SISE and related problems have received considerable attention during the past several decades.

State estimation under unknown inputs is directly related with the SISE problem. In [5], an unbiased minimum variance linear state filter is developed and the state estimation is designed independently with the unknown inputs. The design in [5] is extended to a more general filter structure in [6] and the convergence conditions are given for linear time-invariant systems. The optimization in [5] and [6] has been conducted by employing a linear recursive filter and minimum variance estimation. Moreover, the global optimality of the design of the recursive filters in [5] and [6] is presented in [7] and it is shown that the optimal solution over the class of all linear unbiased estimates can be written in the form of a linear recursive filter. An optimal filter design with stability conditions is proposed in [8] for systems with direct feedthrough. In addition to linear minimum variance estimation, some other methods are used for state estimation with unknown inputs. For instance, in [9–11] and the references therein, full-order and reduced-order state observers with unknown inputs are developed by using matrix manipulations. In [12], a sliding mode observer is...
proposed and the observer convergence is proven. In many applications, such as fault detection and diagnosis, it is desirable to estimate the unknown inputs of a dynamic system. There exist numerous works in this area, for example, see [13–16] and the references therein.

The above-discussed work focus on the estimation of either unknown inputs or system states, but not at the same time. It is much more challenging to simultaneously estimate both inputs and state variables because typically they are inherently interconnected and coupled. In [17], a two-stage Kalman filter and an input filtering technique are combined to achieve joint input and state estimation. An asymptotic input and state estimation scheme is developed in [18] for a class of uncertain systems. By employing a linear matrix inequality (LMI)-based technique, an SISE approach is designed in [19] for systems with a linear part plus a Lipschitz nonlinear term. Most recently, in [20, 21], a set of multi-step recursive filters are proposed to jointly estimate inputs and states by minimizing the error variance for discrete-time linear systems without and with the direct feedthrough, respectively. However, due to the complex structure, the stability analysis of the algorithms proposed in [20, 21] is not given.

In this paper, we develop algorithms to simultaneously estimate the inputs and states for discrete-time linear systems with and without direct feedthrough. The proposed algorithms are built upon the linear minimum variance unbiased estimation. Unlike the other existing approaches such as those in [20, 21], the proposed SISE algorithms have stability properties proven explicitly. Numerical simulation examples are illustrated to show the effectiveness of the theoretical findings. We also compare the performance of the proposed SISE algorithms with other reported algorithms. The results show comparable performance among these algorithms but the proposed ones in this paper have guaranteed estimation stability.

The remainder of the paper is organized as follows. In Section 2, we briefly formulate the problem. Section 3 presents the development of the SISE algorithm for systems with direct feedthrough, and the stability analysis of the algorithm is presented in Section 4. Section 5 extends the results to the systems without direct feedthrough. Section 6 gives illustrative examples to demonstrate the effectiveness of the proposed algorithms and performance comparison with existing SISE algorithms. Finally, we conclude the paper in Section 7.

2. PROBLEM FORMULATION

The notation used throughout the paper is as follows. Small case letters denote vectors and capital letters denote matrices. For matrix $X$, we use $X^T$ and $X^{-1}$ to indicate the transpose and inverse of $X$, respectively. For random variables, ‘E’ indicates expectation. We use $\text{det}(X)$ to denote the determinant of $X$. For symmetric matrix, $X \succ 0$ or $X \succeq 0$ indicates that $X$ is positive definite or nonnegative definite, respectively, and $X \succeq Y$ indicates $X - Y \succeq 0$. We use $\| \cdot \|_2$ and $\| \cdot \|_\infty$ to denote the 2-norm and $\infty$-norm, respectively.

We consider a discrete-time linear system with direct feedthrough as shown in Figure 1

$$
\begin{align*}
x_{k+1} &= Ax_k + Bu_k + w_k, \\
y_k &= Cx_k + Du_k + v_k,
\end{align*}
$$

where $x_k \in \mathbb{R}^n$ denotes the system state variable at time instant $k$, $u_k \in \mathbb{R}^m$ is the unknown input and $y_k \in \mathbb{R}^p$ the system measurement. $A$, $B$, $C$ and $D$ are known system matrices with compatible dimensions. The process noise $w_k$ and measurement noise $v_k$ are assumed to be mutually
uncorrelated zero-mean white noises with known covariances, namely
\[ \mathbb{E}\{w_k w_\ell^T\} = R_w \delta_{k,\ell}, \quad \mathbb{E}\{v_k v_\ell^T\} = R_v \delta_{k,\ell}, \quad \mathbb{E}\{w_k v_\ell^T\} = 0, \]
where \( \delta_k \) is the Kronecker delta function, \( R_w > 0 \) and \( R_v > 0 \) are variances of \( \{w_k\} \) and \( \{v_k\} \), respectively. Throughout the paper, it is assumed that \( D \) is of full column rank. As will be shown in Lemma 1, this is necessary to ensure unbiased estimation.

From the observer theory for deterministic linear systems [22], we design the input and state estimators, respectively, as follows:
\[
\hat{u}_k = H_k (y_k - C \hat{x}_k), \quad \hat{x}_{k+1} = A \hat{x}_k + B \hat{u}_k + L_k (y_k - C \hat{x}_k - D \hat{u}_k),
\]
where \( \hat{u}_k \) represents the input estimate and \( \hat{x}_k \) the state estimate. \( H_k \) and \( L_k \) are the gain matrices that will be determined later. As indicated in (1), \( y_k \) is the first measurement that contains information of \( u_k \), so \( \hat{u}_k \) can be reconstructed from the residual between \( y_k \) and \( C \hat{x}_k \). In addition, the fact that \( x_{k+1} \) depends on \( x_k \) and \( u_k \) motivates \( \hat{x}_{k+1} \) to be recovered by \( \hat{x}_k \) and \( \hat{u}_k \) with a correcting term.

The covariance matrices of the estimation errors are defined as
\[
P^u_k = \mathbb{E}\{\hat{u}_k \hat{u}_k^T\}, \quad P^{ux}_k = \mathbb{E}\{\hat{u}_k \hat{x}_k^T\}, \quad P^x_{k+1} = \mathbb{E}\{\hat{x}_{k+1} \hat{x}_{k+1}^T\},
\]
where \( \hat{u}_k = u_k - \hat{u}_k \) and \( \hat{x}_k = x_k - \hat{x}_k \). It is straightforward to see that matrices \( P^u_k \) and \( P^x_{k+1} \) are symmetric positive definite (SPD).

The objectives of this paper are summarized as follows:

**P1.** To design the input and state estimators, that is, determining estimator gains \( H_k \) and \( L_k \) in (2) and (3).

**P2.** To analyze the stability properties of the proposed estimation algorithm.

**P3.** To extend the results obtained above to discrete-time linear systems without direct feedthrough.

### 3. The Algorithm Description

This section focuses on the development of the SISE algorithm. We start with the minimum variance unbiased design, that is, minimizing \( P^u_k \) to derive \( L_k \), and minimizing \( P^x_{k+1} \) to obtain \( H_k \). Then we present a numerically stable algorithm to compute the design.

#### 3.1. Preliminaries

To study the optimality property of the proposed SISE algorithm, it is important to ensure that the estimates are unbiased. We have the following results.

**Lemma 1**
For unbiased input and state estimates given in (2) and (3), matrix \( D \) of system (1) must be of full column rank, and the following initial condition must be satisfied:
\[
\hat{x}_0 = \mathbb{E}(x_0).
\]

**Proof**
Substituting the state equations (1) into (2) and (3), respectively, we obtain
\[
\hat{u}_k = -H_k (C \hat{x}_k + v_k) + (I - H_k D) u_k,
\]
\[
\hat{x}_{k+1} = (A - L_k C) \hat{x}_k + (B - L_k D) \hat{u}_k - L_k v_k + w_k,
\]
where \( I \) is the identity matrix. Recursively applying the above dynamics until \( k = 0 \), it can be seen that the estimates are unbiased, namely, \( \mathbb{E}(\hat{u}_k) = 0 \) and \( \mathbb{E}(\hat{x}_k) = 0 \), if both (5) and the following input unbiasedness constraint are satisfied
\[
H_k D = I.
\]
From constraint (8), \( D \) should be of full column rank. Proof of Lemma 1 is completed. \( \square \)
Under the above unbiasedness condition in Lemma 1, (6) is then simplified as
\[ \tilde{u}_k = -H_k (C \tilde{x}_k + v_k). \] (9)

### 3.2. Input estimation

The following theorem provides the design of optimal $H_k^*$ that leads to minimum variance of input estimation. That is, with the designed $H_k^*$, we will have the lowest possible $P^u_k$.

**Theorem 1**

Assume that the input estimation given in (2) is unbiased for system (1). If the optimal gain matrix $H_k^*$ is
\[ H_k^* = (D^T Q_k^{-1} D)^{-1} D^T Q_k^{-1}, \] (10)
where $Q_k = C P^x_k C^T + R_v$, then the covariance of input estimation error $P^u_k$ is minimized.

**Proof**

Rewrite $P^u_k$ by plugging (9) into (4) as follows:
\[ P^u_k = H_k Q_k H_k^T. \] (11)

It is desirable to design $H_k$ such that $P^u_k$ is minimum under the constraint (8). Consider using the Lagrange multipliers with the constraint (8). Let us introduce the Lagrange multiplier $\Lambda$, which is a symmetric matrix of appropriate dimension. The Lagrange function $P^u_k(\Lambda)$ is defined as

\[ P^u_k(\Lambda) = H_k Q_k H_k^T + (1 - H_k D)\Lambda + \Lambda^T (1 - D^T H_k^T) \]
\[ = (H_k - \Lambda^T D^T Q_k^{-1}) Q_k (H_k - \Lambda^T D^T Q_k^{-1})^T \]
\[ - [\Lambda - (D^T Q_k^{-1} D)^{-1} (D^T Q_k^{-1} D) [\Lambda - (D^T Q_k^{-1} D)^{-1}] + (D^T Q_k^{-1} D)^{-1}]. \] (12)

To achieve minimum $P^u_k(\Lambda)$, which is SPD, the first term on the right-hand side should make zero contributions (not negative contributions because $Q_k$ is SPD). This occurs only if
\[ H_k = \Lambda^T D^T Q_k^{-1}. \]

Combining (12) with (8) to solve for $\Lambda$, we have
\[ \Lambda = (D^T Q_k^{-1} D)^{-1}. \]

Then the optimal $H_k^*$ is obtained as in (10), and the minimum $P^u_k$ is given by
\[ P_k^u = H_k^* Q_k H_k^{*T} = (D^T Q_k^{-1} D)^{-1}. \] (13)

This proves the theorem. \[ \square \]

Theorem 1 achieves minimization of $P^u_k$ by determining the optimal $H_k^*$. That is, for any $H_k \neq H_k^*$ satisfying the constraint (8) and with (11) and (13), we have
\[ H_k Q_k H_k^T \geq (D^T Q_k^{-1} D)^{-1}. \]

In addition, consider two matrices $X \geq Y \geq 0$. Then the eigenvalues of $X$ will be less than those of $Y$ if arranged in the same order [23], implying the trace of $X$ is larger than that of $Y$. Hence, if $P^u_k$ is minimized, then its trace, i.e. the mean square estimation error, is also minimized [6].

With the optimal $H_k^*$ and from (4) and (9), we have
\[ P^u_k^{ex} = E(-H_k^* [C \tilde{x}_k + v_k] \tilde{x}_k^T) = -H_k^* C P^x_k. \] (14)
3.3. State estimation

For state estimation, we similarly aim to design the optimal $L_k^*$ by minimizing the covariance of state estimation error $P_{k+1}^x$.

Define the following matrices:

$$
S_k = M O_k M^T, \quad T_k = M O_k N^T - B H_k^* R_v, \quad (15a)
$$

$$
U_k = N O_k N^T + R_v - D H_k^* R_v - R_v H_k^T D^T, \quad (15b)
$$

where

$$
M = [A \ B], \quad N = [C \ D], \quad O_k = \begin{bmatrix}
P_k^x & (P_{k}^{ux})^T \\
P_k^{ux} & P_k^u
\end{bmatrix}. \quad (15c)
$$

Before proceeding to obtain the optimal gain matrix $L_k^*$, we show the following property for $U_k$.

**Lemma 2**

For any $k$, $U_k$ in (15b) is singular, that is, $\det U_k = 0$.

**Proof**

Expanding $U_k$ gives

$$
U_k = (I - D H_k^*)(C P_k^x C^T + R_v)(I - D H_k^*)^T
$$

from which it follows that

$$
\det U_k = \det(I - D H_k^*) \det(C P_k^x C^T + R_v) \det(I - D H_k^*)^T.
$$

Because $\det(I - D H_k^*) = \det(I - H_k^* D) = 0$, we have $\det U_k = 0$.

From (4) and (7), we have

$$
P_{k+1}^x = S_k - L_k T_k^T - T_k L_k^T + L_k U_k L_k^T + R_v. \quad (16)
$$

Note that, however, the matrix $U_k$ is singular by Lemma 2. Therefore, there exists no unique $L_k^*$ that achieves minimization of $P_{k+1}^x$. However, we can assume that $U_k$ is nonsingular to see some quick results. Then

$$
P_{k+1}^x = (L_k - T_k U_k^{-1}) U_k (L_k - T_k U_k^{-1})^T + S_k - T_k U_k^{-1} T_k^T + R_v.
$$

The optimal $L_k^*$ that yields lowest possible $P_{k+1}^x$ is given by

$$
L_k^* = T_k U_k^{-1}.
$$

To avoid the singularity problem to obtain a unique gain matrix with guaranteed stability of the SISE estimation, we seek for an alternative construction of related matrices as follows:

$$
\hat{S}_k = M \hat{O}_k M^T, \quad \hat{T}_k = M \hat{O}_k N^T, \quad \hat{U}_k = N \hat{O}_k N^T + R_v, \quad \hat{H}_k^* = (D^T \hat{Q}_k^* D)^{-1} D^T \hat{Q}_k^* \quad (17a)
$$

$$
\hat{Q}_k = C \hat{P}_k^x C^T + R_v \quad \hat{O}_k = \begin{bmatrix}
\hat{P}_k^x & (\hat{P}_k^{ux})^T \\
\hat{P}_k^{ux} & \hat{P}_k^u
\end{bmatrix}, \quad \hat{P}_k^{ux} = -\hat{H}_k^* \hat{P}_k^x, \quad \hat{P}_k^u = \hat{H}_k^* \hat{Q}_k^* \hat{H}_k^T \quad (17b)
$$

Note that

$$
\hat{P}_{k+1}^x = (\hat{L}_k - \hat{T}_k \hat{U}_k^{-1}) \hat{U}_k (\hat{L}_k - \hat{T}_k \hat{U}_k^{-1})^T + \hat{S}_k - \hat{T}_k \hat{U}_k^{-1} \hat{T}_k^T + R_v.
$$

If the near-optimal matrix gain $L_k^*$ is chosen as

$$
\hat{L}_k = \hat{T}_k \hat{U}_k^{-1} \quad (18)
$$

then $\hat{P}_{k+1}^x$ will achieve its minimum

$$
\hat{P}_{k+1}^x = \hat{S}_k - \hat{T}_k \hat{U}_k^{-1} \hat{T}_k^T + R_v = \hat{S}_k - \hat{L}_k \hat{T}_k^T + R_v. \quad (19)
$$
3.4. Algorithm summary

A summary of the proposed procedures is given in Algorithm 1:

**Algorithm 1**: The SISE algorithm for discrete-time linear systems with direct feedthrough.

Initialize: $\hat{x}_0 = \mathbb{E}(x_0)$, $\hat{P}_0 = p_0I$, where $p_0$ is a large positive value;

for $k = 0$ to $N-1$ do

\[ \hat{Q}_k = C \hat{P}_k C^T + R_v; \]

\[ \hat{H}_k^* = (D^T \hat{Q}_k^{-1} D)^{-1} D^T \hat{Q}_k^{-1}, \quad \hat{u}_k = \hat{H}_k^*(y_k - C\hat{x}_k); \]

\[ \hat{P}_k^u = \hat{H}_k^* \hat{Q}_k \hat{H}_k^{*T}; \]

if $k < N - 1$ then

\[ \tilde{\hat{P}}^u_k = -\hat{H}_k^* C \hat{P}_k^x, \]

\[ \hat{O}_k = \begin{bmatrix} \tilde{\hat{P}}^u_k & (\tilde{\hat{P}}^u_k)^T \end{bmatrix}; \]

\[ \hat{S}_k = [A\, B][\hat{O}_k \, A\, B]^T, \quad \hat{T}_k = [A\, B][\hat{O}_k \, C\, D]^T, \quad \hat{U}_k = [C\, D][\hat{O}_k \, C\, D]^T + R_v; \]

\[ \hat{L}_k^x = \hat{T}_k \hat{U}_k, \quad \hat{x}_{k+1} = A\hat{x}_k + B\hat{u}_k + \hat{L}_k^x[y_k - C\hat{x}_k - D\hat{u}_k]; \]

\[ \hat{P}_k^x = \hat{S}_k - \hat{L}_k^x \hat{T}_k^T + R_w \]

end

end

Remark 1

First, since $U_k$ is singular, we will obtain no numerically feasible algorithm if continuing to follow the standard procedures to use $L_k^+ = \hat{T}_k U_k^{-1}$. As a result, it is imperative to restructure the matrices to ensure numerical feasibility. Second, further study on the singularity of $U_k$ shows that it is caused by the correlation between $\hat{u}_k$ and $v_k$. If the correlation is zero, then the singularity problem will not occur. That is actually how we restructure the matrices. To see this, if the correlation is ignored, the matrices in (17a) are equivalent to their original counterparts in (14) and (15c). Finally, we believe that the reconstruction is appropriate because of the guaranteed stability of the algorithm that will be proven in Section 4.

Remark 2

The development of Algorithm 1 is based on unbiased minimum variance estimation (MVUE), which has been commonly used to address state estimation problems, e.g. [5, 6, 20, 21, 24]. However, in order to obtain an SISE algorithm with guaranteed stability, we design Algorithm 1 by taking a tradeoff between optimality (minimum variance) and stability due to an aforementioned structural singularity property of the matrix $U_k$. Specifically, optimality is sacrificed slightly, but at the same time, the algorithm will gain stability properties that can be proven rigorously. We will note in Section 6 that Algorithm 1, though not being optimal, produces satisfactory and comparable performance with other existing SISE algorithms without guaranteed stability in design.

Remark 3

Consider the case of a linear time-varying discrete-time system

\[ x_{k+1} = A_k x_k + B_k u_k + w_k, \]

\[ y_k = C_k x_k + D_k u_k + v_k, \]

where $A_k$, $B_k$, $C_k$ and $D_k$ may change with time and the properties of the noises $w_k$ and $v_k$ follow the assumptions in Section 2. It is straightforward to extend Algorithm 1 to such systems by replacing $A$, $B$, $C$ and $D$ with their time-varying counterparts $A_k$, $B_k$, $C_k$ and $D_k$. The derivation process is analogous to the discussion in this section and we omit it here.
4. STABILITY ANALYSIS

We define the stability of the SISE algorithms in this paper as follows.

Definition 1

The sequences \( \{P_k^x\} \) and \( \{P_k^u\} \) are said to be stable if there exists \( \bar{P}^x < \infty \) and \( \bar{P}^u < \infty \) such that

\[
\lim_{k \to \infty} P_k^x = \bar{P}^x, \quad \lim_{k \to \infty} P_k^u = \bar{P}^u.
\]

Accordingly, the SISE algorithm is stable if both covariance matrices \( \{P_k^x\} \) and \( \{P_k^u\} \) are stable.

The major idea to prove the stability of the proposed SISE Algorithm 1 is sketched as follows. By inspecting Algorithm 1, it is found that its stability depends on the stability of \( P_k^x \) since \( P_k^u \) is a monotonically increasing function of \( P_k^x \). Thus the stability analysis of Algorithm 1 is reduced to the analysis of the upper boundedness of \( P_k^x \) as \( k \to \infty \). To analyze the upper boundedness of \( P_k^x \), it is necessary to analyze the convergence properties of \( \hat{P}_{k+1}^x \). We first formulate the matrix \( \hat{P}_{k+1}^x \) calculation as a solution of a Riccati-like matrix equation. We then utilize and extend some results in convergence analysis of solutions of the Riccati equation and its variants in [25–28] to obtain the stability conditions for our algorithm. We now present such a development in details.

From (17a) and (19), \( \hat{P}_{k+1}^x \) is expressed as

\[
\hat{P}_{k+1}^x = M\hat{O}_k M^T - M\hat{O}_k N^T (N\hat{O}_k N^T + R_v)^{-1} N\hat{O}_k M^T + R_w.
\]

We define a generalized algebraic Riccati (GAR) function \( g(X) \) as follows:

\[
g(X) = M\hat{O}(X) M^T - M\hat{O}(X) N^T (N\hat{O}(X) N^T + R_v)^{-1} N\hat{O}(X) M^T + R_w,
\]

where \( \hat{O}(X) \) has the same structure as \( \hat{O}_k \) in (17a) with \( \hat{P}_k^x \) replaced by \( X \), that is

\[
\hat{O}(X) = \begin{bmatrix} X & (\hat{P}_k^x)^T \\ \hat{P}_u & \hat{P}_u \end{bmatrix}
\]

with

\[
\hat{P}_u = [D^T (CXC^T + R_u)^{-1} D]^{-1}, \quad \hat{P}_u = [D^T (CXC^T + R_u)^{-1} C]^{-1}.
\]

Similar to the SPD requirement of \( \hat{P}_k^x \) in \( \hat{O}_k \), we assume that \( X \) is also SPD in \( \hat{O}(X) \). By (22), (21) can be written as an iterative equation

\[
\hat{P}_{k+1}^x = g(\hat{P}_k^x).
\]

We define a Riccati operator

\[
\phi(K, X) = F\hat{O}(X) F^T + V,
\]

where \( F = M + KN \), \( V = KR_u K^T + R_w \) for matrix \( K \) with compatible dimensions. The following theorem establishes the convergence properties of the GAR function \( g(X) \).

Theorem 2

Assume that

(A1) there exist a \( \tilde{K} \) and a \( \bar{P} > 0 \) such that \( \bar{P} > \phi(\tilde{K}, \bar{P}) \).

Then, for any \( \bar{P}_0 > 0 \), the sequence \( \{\hat{P}_k^x\} \) generated by the iterative equation \( \hat{P}_{k+1}^x = g(\hat{P}_k^x) \) converges, namely

\[
\lim_{k \to \infty} \hat{P}_k^x = \bar{P},
\]

where \( \bar{P} \) is the fixed point of the GAR function \( g(X) \), that is, \( \bar{P} = g(\bar{P}) \).

Proofs of Theorems 2 and 3 with related lemmas are given in Appendix A.
Remark 4
The convergence properties of Algorithm 1 can be quickly concluded from Theorem 2, which presents a sufficient condition (A1) that guarantees the convergence of \( \hat{P}_x^k \). If (A1) is satisfied, then \( \hat{P}_x^k \) converges to the fixed point of the GAR function \( g(X) \), regardless of the initial \( \hat{P}_0 \). However, with the convergence properties of \( \hat{P}_x^k \) proven, we still need to explore the upper boundedness of the true covariances, \( P_x^k \) and \( P_u^k \).

Suppose \( \hat{P}_x^k \rightarrow \overline{P} \) as \( k \rightarrow \infty \) by Theorem 2. As a consequence, \( \hat{L}_k^x \) and \( \hat{H}_k^x \) converge to fixed values, denoted as \( \overline{L} \) and \( \overline{H} \), respectively. The following theorem can be established.

Theorem 3
Assume that
(A2) there exists \( 0 < \tilde{X} < \infty \) such that \( \tilde{X} > \varphi(\tilde{X}) \), where
\[
\varphi(\tilde{X}) = (\overline{LN} - M) \hat{O}(\tilde{X})(\overline{LN} - M)^T.
\]
Then, for any initial value \( P_0 \geq 0 \), the sequences \( \{P_x^k\} \) and \( \{P_u^k\} \) are upper bounded as \( k \rightarrow \infty \).

Remark 5
Based on results in Theorem 2, Theorem 3 shows that when (A2) is satisfied, the true error covariances, \( P_x^k \) and \( P_u^k \) have upper bounds. In simulation, it is observed that \( P_x^k \) is also convergent to a fixed value. Analysis of the exact convergence properties of \( P_x^k \) is one of the on-going research directions and we will report new development in the future.

Remark 6
Stability properties of the classical estimation techniques the Kalman filter are often formulated in terms of some stabilizability/detectability conditions. However, considered in this work is the more complicated SISE problem, which leads to the analysis of a non-standard Riccati equation. We found that stabilizability/detectability are not sufficient to guarantee stability in this case. It is also inferred intuitively that such conditions are implicitly included as part of Theorems 2 and 3. Therefore, it is left as an on-going research topic.

Remark 7
It should be noted that the stability analysis developed for linear time-invariant systems here is not considered generalizable to linear time-varying systems, although Remark 3 claims that Algorithm 1 can be extended to the time-varying case.

5. EXTENSION TO DISCRETE-TIME LINEAR SYSTEMS WITHOUT DIRECT FEEDTHROUGH

The previous sections focused on SISE for discrete-time linear systems with direct feedthrough. In this section, we will obtain some analogous results by applying the techniques to systems with no direct feedthrough.

Consider the discrete-time linear system without direct feedthrough
\[
\begin{align*}
    x_{k+1} & = Ax_k + Bu_k + w_k, \\
    y_k & = Cx_k + v_k. \\
\end{align*}
\] (25)

Obviously, (25) is a special case (1) by setting \( D = 0 \). Since no feedthrough is applied here, the input and state estimators are designed, respectively, as follows:
\[
\begin{align*}
    \hat{u}_k & = H_k(y_{k+1} - CAx_k), \quad (26) \\
    \hat{x}_{k+1} & = A\hat{x}_k + Bu_k + L_k(y_k - C\hat{x}_k). \quad (27)
\end{align*}
\]
Following similar lines in Section 3, we can readily derive the SISE algorithm for the system (25), which is summarized in Algorithm 2. To ensure that Algorithm 2 has stability properties proven explicitly, the correlation between $\hat{u}_k$ and $w_k$ is assumed to be zero in the derivation.

Algorithm 2: The SISE algorithm for discrete-time linear systems without direct feedthrough.

| Initialize: $\hat{x}_0 = E(x_0)$, $P_0^x = p_0 I$, where $p_0$ is a large positive value; |
| for $k = 0$ to $N - 1$ do |
| $\hat{Q}_k = C \hat{P}_k^x C^T + CR_w C^T + R_v$; |
| $\hat{H}_k^x = (B^T \hat{Q}_k^{-1} CB)^{-1} B^T \hat{Q}_k^{-1}$, $\hat{u}_k = \hat{H}_k^x (y_{k+1} - CA \hat{x}_k)$; |
| $\hat{P}_k^u = \hat{H}_k^x \hat{Q}_k \hat{H}_k^x$; |
| if $k < N - 1$ then |
| $\hat{P}_k^u = -\hat{H}_k^x CA \hat{P}_k^x$; |
| $\hat{O}_k = \begin{bmatrix} \hat{P}_k^u & \hat{P}_k^{ux} \\ \hat{P}_k^{ux} & \hat{P}_k^u \end{bmatrix}$; |
| $\hat{S}_k = [A \, B] \hat{O}_k [A \, B]^T$, $\hat{T}_k = [A \, B] \hat{O}_k [C \, 0]^T$, $\hat{U}_k = [C \, 0] \hat{O}_k [C \, 0]^T + R_v$; |
| $\hat{L}_k^x = \hat{T}_k \hat{U}_k^{-1}$, $\hat{x}_{k+1} = A \hat{x}_k + B \hat{u}_k + \hat{L}_k^x [y_k - C \hat{x}_k]$; |
| $\hat{P}_{k+1}^x = \hat{S}_k - \hat{L}_k^T \hat{T}_k + R_w$ |
| end |

Remark 8
To ensure the unbiasedness of Algorithm 2, two assumptions must be established: (1) the matrix $CB$ is of full column rank, and (2) the initial condition $\hat{x}_0 = E(x_0)$ is satisfied. The rationale for these assumptions is quite similar to the proof of Lemma 1. Note that an equivalent statement of (1) is $\text{rank}(CB) = \text{rank}(B) = m$, which is common in the literature, e.g. see [5, 6, 21].

Not surprisingly, $P_k^x$ again determines the stability of Algorithm 2. Let us consider $\hat{P}_k^x$ in the first place. The propagation of $\hat{P}_k^x$ is governed by

$$\hat{P}_{k+1}^x = M \hat{O}_k M^T - M \hat{O}_k N^T (N \hat{O}_k N^T + R_v)^{-1} N \hat{O}_k M^T + R_w,$$

where $M = [A \, B]$ and $N = [C \, 0]$. The GAR function $g(X)$ for Algorithm 2 is defined as

$$g(X) = M \hat{O}(X) M^T - M \hat{O}(X) N^T (N \hat{O}(X) N^T + R_v)^{-1} N \hat{O}(X) M^T + R_w,$$

where $\hat{O}(X)$ evolves from $\hat{O}_k$ by replacing $\hat{P}_k^x$ with $X$.

It is clear that the GAR functions defined in (29) and (22) have common characteristics. The properties of (22) have been studied extensively in Section 4. In similar fashion, we study (29) for Algorithm 2 to deduce the stability results. Omitting the intermediate proofs, let us present Theorem 4 for the stability analysis of Algorithm 2.

Theorem 4
Assume that
(A3) there exists a $\tilde{K}$ and a $\tilde{P} > 0$ such that $\tilde{P} \succ \phi(\tilde{K}, \tilde{P})$, where

$$\phi(K, X) = (M + K N) \hat{O}(X) (M + K N)^T + K R_v K^T + R_w.$$

Then, for any $P_0 \succeq 0$, the sequence $\{ \hat{P}_k^x \}$ generated by the iterative equation $\hat{P}_{k+1}^x = g(\hat{P}_k^x)$ converges, namely

$$\lim_{k \to \infty} \hat{P}_k^x = \overline{\tilde{P}},$$
where $\hat{P}$ is the fixed point of the GAR function $g(X)$, that is, $\hat{P} = g(\hat{P})$. Define $\hat{L} = \lim_{k \to \infty} \hat{L}_k$ and $\hat{H} = \lim_{k \to \infty} \hat{H}_k$. Further, assume that

(A4) there exists $0 < X < \infty$ such that $X > \varphi(X)$, where

$$\varphi(X) = (\hat{L}N - M)\hat{O}(X)(\hat{L}N - M)^T.$$  

Then, for any initial value $P_0 \geq 0$, the sequences of true covariances, $\{P_k^x\}$ and $\{P_k^u\}$, are upper bounded as $k \to \infty$.

**Remark 9**

Theorem 4 indicates that, when Algorithm 2 is applied to the system (25), the estimation error covariances $P_k^x$ and $P_k^u$ are upper bounded when (A3) and (A4) are satisfied. Note that (A3) guarantees the convergence of $\hat{P}_k$ to a fixed point, and that (A4) further ensures the upper boundedness of $P_k^x$ and $P_k^u$.

### 6. NUMERICAL EXAMPLES

In this section, three numerical examples are presented to illustrate the effectiveness of the proposed algorithm.

**Example 1**

Consider an LTI system described by

$$A = \begin{bmatrix} 0.67 & 0 \\ 0 & 0.53 \end{bmatrix}, \quad B = \begin{bmatrix} 1.00 \\ 0.53 \end{bmatrix}, \quad C = \begin{bmatrix} 0.95 & 0.01 \\ 0.03 & 1.39 \end{bmatrix}, \quad D = \begin{bmatrix} 1.05 \\ 1.20 \end{bmatrix},$$  

$$R_w = R_v = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix}.$$  

In this example, the input $\{u_k\}$ is taken as a Gaussian white noise with zero mean and unit variance. Suppose that no information about $\{u_k\}$ is available for use. For initialization, set $\hat{P}_0 = 10^6I$. In the simulation, only the output $\{y_k\}$ is used and Algorithm 1 is applied. The observed results are summarized as follows:

1. The input estimation and state estimation results are shown in Figure 2. It is observed that input estimates are close to the actual input values. Meanwhile, estimation of the two states also exhibits good performance as there are only small differences between the estimates and their true values.

2. In the simulation, we find that $\hat{P}_k^x$ quickly converges to $\hat{P}$:

$$\hat{P} = \begin{bmatrix} 0.1133 & 0.0027 \\ 0.0027 & 0.0885 \end{bmatrix}.$$  

It is easy to check that $\hat{P} = g(\hat{P})$ holds true, which indicates that $\hat{P}$ is the fixed point of the GAR function $g(X)$. This result confirms the convergence analysis in Section 4. Meanwhile, we note that the true error variance $P_k^x$ also converges to a fixed point $P$:

$$P = \begin{bmatrix} 0.1321 & 0.0113 \\ 0.0113 & 0.0924 \end{bmatrix}.$$  

The trace of $P_k^x$ shown in Figure 3 (solid line) demonstrates the monotonically converging trend. This suggests that, in addition to being upper bounded, $P_k^x$ is likely to have certain convergence property.

3. To illustrate the performance of the proposed algorithm, we compare the results by Algorithm 1 with the Gillijns–De Moor algorithm in [20]. In Figure 3, it is shown that $\text{tr}(P_k^x)$ yielded by
Figure 2. Example 1: Actual values (solid line) vs estimates (dashed line). From top to bottom: input estimation, the first state estimation, and the second state estimation.

Figure 3. Example 1: Trace of $P_k^x$ vs $k$.

the Gillijns–De Moor algorithm (red-dashed line) is slightly smaller than that by Algorithm 1. We also compare the input estimation errors $\tilde{u}_k$ and the state estimation errors $\tilde{x}_k$ under both algorithms. The corresponding 2-norm and $\infty$-norm values of the estimation errors in one implementation are shown in Table I. From Figure 3 and Table I, we conclude that both algorithms produce comparable estimation performance.

**Example 2**

In this example, we study the performance of Algorithm 2 and its comparison with the Gillijns–De Moor algorithm. Now consider the system

$$A = \begin{bmatrix} 0.67 & 0 \\ 0 & 0.53 \end{bmatrix}, \quad B = \begin{bmatrix} 1.00 \\ 0.53 \end{bmatrix}, \quad C = \begin{bmatrix} 0.95 & 0.01 \\ 0.03 & 1.39 \end{bmatrix}, \quad R_w = R_v = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix}. $$
Table I. Example 1: Comparisons between the norms of input and state estimation errors by Algorithm 1 and the Gillijns–De Moor algorithm.

<table>
<thead>
<tr>
<th></th>
<th>Algorithm 1</th>
<th>Gillijns–De Moor algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input estimation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$|\hat{u}_k|_2$</td>
<td>4.1234</td>
<td>4.0540</td>
</tr>
<tr>
<td>$|\hat{u}<em>k|</em>\infty$</td>
<td>1.5504</td>
<td>1.5504</td>
</tr>
<tr>
<td>State estimation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$|\hat{x}_{1k}|_2$</td>
<td>4.5928</td>
<td>4.5624</td>
</tr>
<tr>
<td>$|\hat{x}<em>{1k}|</em>\infty$</td>
<td>1.0000</td>
<td>2.0814</td>
</tr>
<tr>
<td>$|\hat{x}_{2k}|_2$</td>
<td>3.7564</td>
<td>3.5121</td>
</tr>
<tr>
<td>$|\hat{x}<em>{2k}|</em>\infty$</td>
<td>1.0000</td>
<td>1.0452</td>
</tr>
</tbody>
</table>

Figure 4. Example 2: Actual values (solid line) vs estimates (dashed line). From top to bottom: input estimation, the first state estimation, and the second state estimation.

where no direct feedthrough is present. The estimation results by Algorithm 2 are given in Figure 4; the trace and norm comparison results between Algorithm 2 and the Gillijns–De Moor algorithm are shown in Figure 5 and Table II, respectively.

In fact, through numerous simulations for the purpose of comparison, we consistently observe that the performances of both Algorithm 1 in this paper and the Gillijns–De Moor algorithm are comparable. However, an attractive advantage when applying Algorithms 1 and 2 is that the stability can be verified and ensured in advance, while in [20, 21] the stability and convergence analysis are not always guaranteed.

7. CONCLUSION

This paper considered SISE for systems with and without direct feedthrough. The challenge of the SISE problem lies in limited information such as the output measurements and system structure and coupling between input and state estimations. Our major contribution is to obtain SISE algorithms with proven stability properties. We first analyzed the optimality conditions for minimum variance. Then we presented optimal design procedures for the SISE problem. The stability of the proposed algorithms were analyzed and guaranteed under the proposed SISE algorithms. We have shown...
and demonstrated that the estimation error covariances are upper bounded and thus stable. Several numerical examples were given to illustrate the effectiveness of the proposed algorithm. We also compared the performance of the proposed algorithms with other existing SISE algorithms, achieving comparable performance with guaranteed stability. Unknown inputs often appear in networked control systems, which have been an active research area in the past decade [28–31]. It is worthwhile extending the SISE algorithm to networked dynamic systems subject to random delays and packet losses [32].

APPENDIX A

We need following lemmas to prove Theorems 2 and 3.

**Lemma A1**

$\tilde{O}(X)$ is SPD and monotonically increasing with $X$.

**Proof**
The proof is straightforward and thus is omitted.

**Lemma A2**
The following facts hold true for functions $g(X)$ and $\phi(K, X)$:

(a) $g(X) = \phi(K_X, X)$ with $K_X = -M\tilde{O}(X)N^TN\tilde{O}(X)N^T + R_v)^{-1}$;
(b) $g(X) = \min_K \phi(K, X)$; and
(c) for $0 < X \leq Y$, $g(X) \leq g(Y)$.

Table II. Example 2: Comparisons between the norms of input and state estimation errors by Algorithm 2 and the Gillijns–De Moor algorithm.

<table>
<thead>
<tr>
<th></th>
<th>Algorithm 2</th>
<th>Gillijns–De Moor algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input estimation</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$|\tilde{u}_k|_2$</td>
<td>4.0272</td>
<td>4.0343</td>
</tr>
<tr>
<td>$|\tilde{u}<em>k|</em>\infty$</td>
<td>0.9031</td>
<td>0.9050</td>
</tr>
<tr>
<td><strong>State estimation</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$|\tilde{x}_{1k}|_2$</td>
<td>3.8715</td>
<td>3.6163</td>
</tr>
<tr>
<td>$|\tilde{x}<em>{1k}|</em>\infty$</td>
<td>1.0000</td>
<td>1.4749</td>
</tr>
<tr>
<td>$|\tilde{x}_{2k}|_2$</td>
<td>3.0387</td>
<td>2.1247</td>
</tr>
<tr>
<td>$|\tilde{x}<em>{2k}|</em>\infty$</td>
<td>1.0000</td>
<td>0.8100</td>
</tr>
</tbody>
</table>
Proof
The fact (a) can be readily obtained by substituting $K = K_X$ into (24). For (b), it is straightforward to obtain
\[
\phi(K, X) = (M + KN)\hat{O}(X)(M + KN)^T + KR_vK^T + R_w.
\]
We can solve for $K$ that minimizes $\phi(K, X)$ by letting
\[
\frac{\partial \phi(K, X)}{\partial K} = 0.
\]
It is found that $K = K_X$ is the solution of the above equation. For (c), as $\hat{O}(X)$ monotonically increases with $X$, so does $\phi(X)$. Then for $0 < X \leq Y$, we have
\[
g(X) = \phi(K_X, X) \leq \phi(K_Y, X) \leq \phi(K_Y, Y) = g(Y),
\]
which proves the fact. \hfill \Box

Lemma A3 (Sinopoli et al. [28])
Assume that $h(\cdot)$ is a monotonically increasing function. Given two sequences $X_{k+1} = h(X_k)$ and $Y_{k+1} = h(Y_k)$, if $X_1 \geq X_0$, then $X_{k+1} \geq X_k$; if $X_1 \leq X_0$, then $X_{k+1} \leq X_k$; and if $X_0 \leq Y_0$, then $X_k \leq Y_k$, for any $k \geq 1$.

Lemma A4
Let $\psi(X) = F\hat{O}(X)F^T$. Assume that there exists $0 < \bar{X} < \infty$ such that $\bar{X} > \psi(\bar{X})$. Consider $X_{k+1} = \psi(X_k) + \Delta$ with $\Delta \geq 0$ and initial value $X_0 \geq 0$. Then the sequence $X_k$ is upper bounded.

Proof
It is noted that $\psi(X)$ is linear, positive definite and monotonically increasing with $X$. For any $0 < X < \infty$, there exist $m \geq 0$ and $0 < r < 1$ such that $X \leq m\bar{X}$ and $\psi(\bar{X}) < r\bar{X}$. Then it follows from Lemma A3 that
\[
0 \leq \psi^k(X) \leq mr^{k-1}\psi(\bar{X}) \leq mr^k\bar{X},
\]
where
\[
\psi^k(X) = \psi(\psi(\psi(\cdots (\cdots (\psi(X)\cdot\cdots)(\cdot\cdots)(\cdot\cdots)(\psi(X))))
\]
and $\psi^0(X) = X$.

Thus as $k \to \infty$, $mr^k\bar{X} \to 0$, and further, $\psi^k(X) \to 0$. This conclusion indicates that there exists $0 \leq m\Delta \leq m\bar{X} < \infty$ such that
\[
X_k \leq \psi^k(X_0) + \sum_{i=0}^{k-1} \psi^i(\Delta) \leq \left( mX_0 r^k + \sum_{i=0}^{k-1} m\Delta r^i \right) \bar{X} \leq \left( mX_0 + \frac{m\Delta}{1-r} \right) \bar{X} < \infty.
\]
The lemma is proven. \hfill \Box

Proof of Theorem 2
Note that
\[
\tilde{P} > \phi(\tilde{K}, \tilde{P}) = \psi(\tilde{P}) + \tilde{K}R_v\tilde{K}^T + R_w \geq \psi(\tilde{P}).
\]
Then from Lemma A2, we obtain
\[
\tilde{P}_{k+1} = g(\tilde{P}_k) \leq \phi(\tilde{K}, \tilde{P}_k) = \psi(\tilde{P}_k) + \tilde{K}R_v\tilde{K}^T + R_w = \psi(\tilde{P}_k) + \hat{\Delta},
\]
where $\hat{\Delta} = R_v\tilde{K}^T + R_w \geq 0$. Using Lemma A4, we conclude that $\{\tilde{P}_k\}$ is upper bounded.

Next, we shall show the sequence $\{\tilde{P}_k\}$ converges to $\tilde{P}$ from any $\tilde{P}_0 \geq 0$. First, consider the extreme case when $\tilde{P}_0 = 0$. Then
\[
0 = \tilde{P}_0 \leq g(\tilde{P}_0) = \tilde{P}_1,
\]
which implies by Lemmas A2(c) and A3 that
\[
\tilde{P}_0 \leq \tilde{P}_1 \leq \tilde{P}_2 \leq \cdots.
\]
Since this sequence is proven upper bounded, its upper bound, \( \bar{P} \), must be the solution of \( \bar{P} = g(\bar{P}) \), namely, the fixed point of function \( g(X) \).

We now consider the case when \( \bar{P} \geq P \). Define
\[
\dot{K} = -A\bar{P}X(C\bar{P}C^T + R_o)^{-1}, \quad \dot{P} = A\dot{K}C, \quad \dot{\psi}(X) = \dot{F}X\dot{F}.
\]
We obtain
\[
\bar{P} = g(\bar{P}) = \phi(\dot{K}, \bar{P}) > \hat{\psi}(\bar{P}).
\]
Thus from Lemma A4, we obtain
\[
\lim_{k \to \infty} \psi^k(X) = 0 \quad \text{for any } X > 0.
\] (A1)

Since \( \bar{P}_0 > \bar{P} \), it holds that \( \hat{P}_1 = g(\bar{P}_0) > g(\bar{P}) = \bar{P} \), which shows \( \hat{P}_1 > \bar{P} \). Let us consider the sequence \( \{\hat{P}_k^x - \bar{P}\} \). We have
\[
0 < (\hat{P}_{k+1}^x - \bar{P}) = g(\hat{P}_k^x) - g(\bar{P}) = \phi(K\hat{P}_k^x, \hat{P}_k^x) - \phi(\dot{K}, \bar{P}) \leq \phi(\hat{K}, \hat{P}_k^x) - \phi(\dot{K}, \bar{P}) = \psi(\hat{P}_k^x - \bar{P}).
\]
The above inequality, together with (A1), implies that \( \lim_{k \to \infty} \hat{P}_{k+1}^x = \bar{P} \).

Last, if \( 0 < \bar{P}_0 < \bar{P} \), then \( g^k(0) < \hat{P}_{k+1}^x = g^k(\bar{P}_0) < g(\bar{P}) = \bar{P} \) and, thus, if \( k \to \infty \), we have \( \lim_{k \to \infty} \hat{P}_{k+1}^x = \bar{P} \), which shows that the sequence \( \hat{P}_{k+1}^x = g(\hat{P}_k^x) \) is monotonically convergent to \( \bar{P} \), for any \( \bar{P}_0 \geq 0 \). The proof is complete. \( \Box \)

**Proof of Theorem 3:**

From (16) it follows that
\[
P_{k+1}^x = S_k - \hat{L}_k^T\hat{T}_k - T_k\hat{L}_k^T - \hat{L}_k^T\hat{T}_k - I_k\hat{L}_k^T + R_w. \quad \text{As } k \to \infty, \quad \text{we have}
\]
\[
\lim_{k \to \infty} P_{k+1}^x = \lim_{k \to \infty} [M O_k M^T - \bar{L}(M O_k N^T - BHR_v)^T - (M O_k N^T - BHR_v)\bar{L}^T]
\]
\[
+ \bar{L}(O_k N^T + R_v - DHR_v - R_v(DH)^T)\bar{L}^T + R_w
\]
\[
= \lim_{k \to \infty} [\bar{L}(O_k N^T + R_v - DHR_v - R_v(DH)^T)\bar{L}^T + R_w].
\]

where \( \Omega = \bar{L}(BHR_v)^T + (BHR_v)^T\bar{L} + \bar{L}(R_v - DHR_v - R_v(DH)^T)\bar{L}^T + R_w \). Note that there always exists a \( \Xi \geq 0 \) such that \( \Omega \leq \Xi \). Then we have \( \lim_{k \to \infty} P_{k+1}^x \leq \lim_{k \to \infty} \phi(P_k^x) + \Xi \). Using Lemma A4, it can be concluded that \( \lim_{k \to \infty} P_{k+1}^x \) is upper bounded. Since \( P_k^x \) is a monotonically increasing function of \( P_k^x \), \( \lim_{k \to \infty} P_k^x \) is upper bounded as well. \( \Box \)

**REFERENCES**


