MODELING OF CLOSED KINEMATIC CHAINS WITH FLEXIBLE LINKS

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Abstract

A way is presented of modeling complex mechanical systems, and particularly closed kinematic chains, that are comprised of elastic components. The motion of each link is viewed using a moving reference frame and the zero tip deformation constraint is introduced to define the orientation of the reference frames. That analysis is combined with the differential-algebraic formulation and the equation of motion is given in constrained form. We investigate the effect of the shortening of the projection and show that ignoring it can lead to gross inaccuracies.

Introduction

Modeling of manipulators and closed kinematic chains that have flexible links has been considered by many researchers. In order to simplify partial differential equations associated with flexible links some of them used assumed modes technic to derive equations of motion [1]-[4]. Others used discretisation of the links like finite element method [5]-[8], or finite segment modeling combined with Kane's equation formulation [9]. Some of the earliest work was done by Neubauer at. al. [10] where they derived equations of motion by using both assumed modes method and finite element method. They considered slider-crank mechanism and only transverse vibrations of the connecting rod.

Since modeling of flexible dynamical systems is very complex most of the early papers considered only transverse vibrations [1],[4],[6], and/or only one link was flexible [2],[3],[4],[6],[10]. Introduction of powerful computers enables researchers to consider both axial and transverse vibrations, and allow all links to be flexible [5],[7],[8],[9].

In this paper, we consider dynamical systems that are comprised of elastic linkages. We write the equations of motion using constrained generalized coordinates and we describe the elastic motion of the linkages in terms of moving coordinate systems. Most of the papers [1], [3]-[10] neglect coupling effect between axial and transverse vibrations (i.e,. shortening of the projection). Therefore, we analyze the characteristics of the response as a function of the shortening of the projection and variety of other factors.

Using constrained generalized coordinates to derive the equations of motion is more suitable than using a set of independent generalized coordinates in a variety of circumstances. For example, in a closed kinematic chain it is difficult to obtain a set of independent generalized coordinates and to express the remaining coordinates in terms of the independent coordinates [10]. Barring a few exceptions, the same situation is encountered even when a method that can handle nonholonomic constraints by means of generalized speeds is used.

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To derive the describing equations in terms of constrained coordinates, one makes use of Lagrange multipliers. The resulting equations are said to be in *differential-algebraic* form [12–16]. These equations are differential equations with respect to the generalized coordinates and algebraic with respect to the Lagrange multipliers, as the derivatives of the Lagrange multipliers do not appear.

After the describing equations are derived, one can reduce these equations into unconstrained form by eliminating the Lagrange multipliers, or deal with these equations in their differential-algebraic form. The latter choice has become possible in the last 25 years, as more analysis has been conducted on differential-algebraic systems, and software has been developed to integrate such equations. In the last ten years, the control of systems described by differential-algebraic equations has begun to see renewed interest [12],[17–19].

The motion of flexible bodies undergoing combined rigid and elastic motion can be analyzed by viewing the motion from a moving reference frame. The use of such a frame imposes a constraint on the motion viewed from this frame. This constraint is usually taken into consideration by selecting the trial functions that describe the motion viewed from the moving reference frame. Two methods, albeit not very suitable for multilink systems, are the rigid body mode and the zero slope constraints [10], [20].

In this paper, we use the zero tip deformation constraint [10], as it is more suitable when dealing with miltilink mechanisms. We show that use of this constraint results in simpler mathematical models and it does not lead to mathematical inconsistencies.

We combine in this paper the differential-algebraic approach with the zero tip deformation constraint, and we derive the describing equations of a closed kinematic chain with elastic components. We integrate these equations and analyze them for accuracy, relevant terms in the equations of motion, the shortening of the projection and controllability.

The Zero Tip Deformation Constraint

Consider the motion of the beam in Fig. 1. The beam is attached to some other body by a pin joint. As the beam moves, it undergoes large-angle rigid as well as elastic motion. When the elastic deformation is small, we can view the motion of the beam as a superposition of a *primary motion* and a *secondary motion*. The primary motion defines a reference frame from which the motion of the beam is observed. The secondary motion is the motion amplitude of the beam as observed from the reference frame. Denoting the reference frame associated with the secondary motion by xyz, and the angle that the x axis makes with an inertial X axis by θ, the position of a point on the beam axis can be represented by

$$\mathbf{r}(x,t) = \mathbf{r}_o(t) + x\mathbf{i} + u(x,y)\mathbf{i} + v(x,t)\mathbf{j} \tag{1}$$

in which $\mathbf{r}_o(t)$ denotes the origin of the reference frame, selected here for convenience as the pin joint. The deformation in the x and y directions is denoted by $u(x,t)$ and $v(x,t)$. The secondary motion is described by the last two terms in the above equation.

There are several ways of selecting the location and orientation of the reference frame associated with the primary motion. The most important consideration is to have the reference frame such that the secondary motion is small and it can be modeled using linear vibration theory. As can be observed from Fig. 1, it is possible to find several orientations of the primary motion (for example x, x', or x'') such that the secondary motion has small amplitudes.

Note that use of the reference frame introduces additional variables to describe the motion and it creates a redundancy [20]. This redundancy is usually dealt with by selecting the reference frame such that a set of constraints are applied to the secondary motion. These effect of these constraints is to alter the boundary conditions associated with the secondary motion. By selecting the expansion functions to describe the secondary motion, these constraints can be satisfied. This, in essence, is tantamount to selecting a set of independent generalized coordinates. It should be noted that these constraints are at a mathematical level, and that they do not change the force and moment balances at the boundaries.

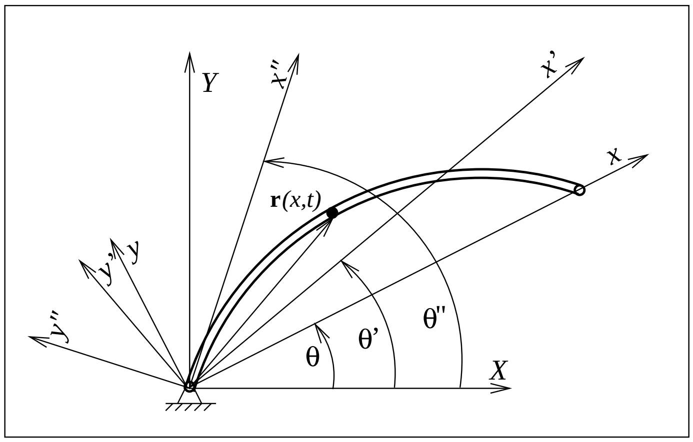

Figure 1: A Flexible Link

In this paper we only consider plane motion. Before describing the zero tip deformation constraint, we discuss two other constraints. The first is known as the *rigid body mode constraint* and the coordinate system $x'y'$ in Fig. 1 describes the secondary motion. This constraint forces the secondary motion to have no rigid body components. For the pinned-free beam under consideration this constraint can be expressed as

$$\int_{body} x^{'} v^{'} \left(x^{'}, t\right) dm = 0 \tag{2}$$

where dm is the differential mass element. We note that $x^{'}$ denotes the rigid body mode. One can then expand the secondary motion using the boundary conditions of a pinned-free beam. The eigenfunctions of a pinned-free beam or orthogonalized polynomials constitute possible choices for expanding the trial functions.

There are a number of disadvantages associated with this formulation. First, one has to use the real-time measurements of the absolute motion of points on the beam and then invoke Eq. (2) to calculate the orientation of the reference frame. If the beam is attached to another member at its other end $\left(x^{'} = L\right)$, then we do not have a pinned-free beam and the concept of rigid body mode is no longer meaningful. For such a case, using the eigenfunctions of a pinned-free beam as trial functions leads to physical incompatibilities, as these trial functions won't be able to satisfy the force balance at $x^{'} = L$. Finally, eigenfunctions of a pinned-free beam are comprised of hyperbolic sines and cosines, which are not very desirable to use from a computational perspective.

The second constraint is known as the *zero slope constraint* and it is implemented by selecting the reference frame $x''y''$ in Fig. 1, such that the secondary motion has zero slope at the pinned end ($x''=0$). One can use the eigenfunctions of a fixed-free beam or the polynomials x''^2, x''^3, ... as trial functions to expand the elastic motion.

The use of this constraint has the advantage that it is very simple to orient the reference frame. However, use of this constraint leads to inaccurate models even when the elastic behavior of the beam is relatively small. Further, all the concerns expressed above when the end $x''=L$ is not free and the convergence of the trial functions are valid for this constraint as well [10].

Let us next consider the *zero tip deformation constraint*, which is invoked by drawing a straight line between the ends $x=0$ and $x=L$. The configuration is represented by the coordinate system xy in Fig. 1. As a result, the boundary conditions on the secondary motion become those of a pinned-pinned beam, so that one can use simple sine functions to expand the secondary motion.

Use of this constraint does not have any of the problems associated with the previous two constraints. It is easy to orient the reference frame, both mathematically and in real time, the trial functions used to expand the secondary motion have no numerical or sensitivity problems, and the force balances at the boundaries are always satisfied, regardless of whether an end is free or connected to another body.

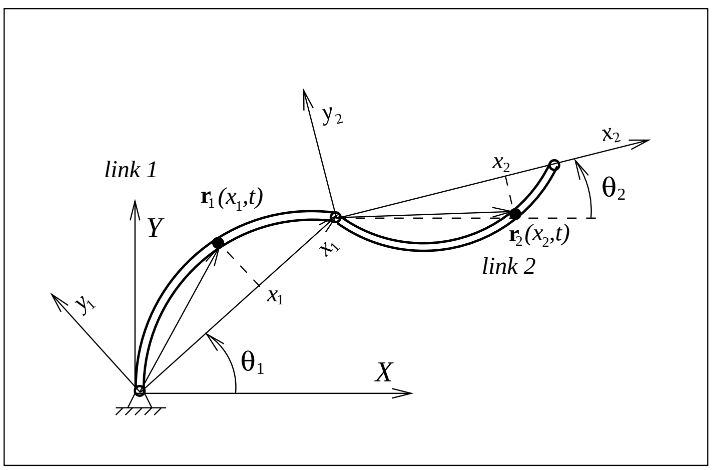

Figure 2: The Two Link Flexible Mechanism

There are two other advantages associated with using the zero tip deformation constraint: i) the resulting equations are simpler than when the other constraints are used, which makes simulating them easier, and ii) the resulting equations have a hierarchical form, which makes it easier to conduct a sensitivity analysis. To demonstrate these advantages, consider the two link flexible mechanism, shown in Fig. 2. For illustrative purposes, we neglect axial stretch, and consider deformation in one direction only. The position vectors for points on the two beams can be written as

$$\mathbf{r}_1(x_1,t)=x_1\mathbf{i}_1+v_1(x_1,t)\mathbf{j}_1, \quad \mathbf{r}_2(x_2,t)=\mathbf{r}_1(L_1,t)+x_2\mathbf{i}_2+v_2(x_2,t)\mathbf{j}_2 \tag{3}$$

where the subscripts 1 and 2 denote the number of the links. The above form for the position vector is valid for either constraint used. The difference arises in the evaluation of the $\mathbf{r}_1(L_1, t)$ term. We expand the secondary motions of the two beams as

$$v_1(x_1, t) = \sum_{k=1}^{n_1} \phi_{1k}(x_1) q_{1k}(t), \quad v_2(x_2, t) = \sum_{k=1}^{n_2} \phi_{2k}(x_1) q_{2k}(t) \tag{4}$$

where ϕ_{1k} and ϕ_{2k} are suitable trial functions, and q_{1k} and q_{2k} are generalized coordinates.

Noting that the angular velocity of the first frame is $\dot{\theta}_1(t)$, we can write $\mathbf{r}_1(L_1, t)$ and its time derivative as

$$\begin{aligned} \mathbf{r}_1(L_1, t) &= L_1 \mathbf{i}_1 + \sum_{k=1}^{n_1} \phi_{1k}(L_1) q_{1k}(t) \mathbf{j}_1 \\ \dot{\mathbf{r}}_1(L_1, t) &= \sum_{k=1}^{n_1} \phi_{1k}(L_1) \dot{q}_{1k}(t) \mathbf{j}_1 + L_1 \dot{\theta}_1(t) \mathbf{j}_1 - \dot{\theta}_1(t) \sum_{k=1}^{n_1} \phi_{1k}(L_1) q_{1k}(t) \mathbf{i}_1 \end{aligned} \tag{5}$$

When using the zero tip deformation constraint $\phi_{1k}(L_1) = 0$, $(k = 1, 2, ...)$ Eqs. (5) reduce to

$$\mathbf{r}_1(L_1, t) = L_1 \mathbf{i}_1, \qquad \dot{\mathbf{r}}_1(L_1, t) = L_1 \dot{\theta}_1(t) \mathbf{j}_1 \tag{6}$$

which are much simpler expressions than the general terms in Eqs. (5). Hence, the expression for $\mathbf{r}_2(x_2, t)$ becomes

$$\mathbf{r}_2(x_2, t) = L_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + v_2(x_2, t) \mathbf{j}_2 \tag{7}$$

Extension of this to a multibody mechanism (two or three dimensional), we can write the deformation of the *p*th element as

$$\mathbf{r}_p(x_p, t) = L_1 \mathbf{i}_1 + L_2 \mathbf{i}_2 + ... + L_{p-1} \mathbf{i}_{p-1} + x_p \mathbf{i}_p + \sum_{k=1}^{n_p} \phi_{pk}(x_p) q_{pk}(t) \mathbf{j}_p \tag{8}$$

Hence, in the expression for the elastic deformation of each element there is only one summation for the secondary motion. Further, as one changes the number of terms that are used for the secondary motion, the changes to the equations of motion can be observed more clearly than in other methods. Note also that the same trial functions are used for all links. A comparison of the zero tip deformation constraint and the zero slope and rigid body mode constraints is conducted in [10]. Some of the advantages of the zero tip deformation constraint disappear when the axial deformation is included in the formulation. In that case, positions of particles of the links are defined as

$$\mathbf{r}_1(x_1, t) = x_1 \mathbf{i}_1 + u_1(x_1, t) \mathbf{i}_1 + v_1(x_1, t) \mathbf{j}_1 \tag{9}$$

$$\mathbf{r}_2(x_2, t) = \mathbf{r}_1(L_1, t) + (x_2 + u_2(x_2, t)) \mathbf{i}_2 + v_2(x_2, t) \mathbf{j}_2 \tag{10}$$

Kinetic Energy for Elastic Links

In this section we investigate the kinetic energy associated with interconnected elastic linkages. The axial deformation is also included in the formulation. The velocities of points on the links are

$$\dot{\mathbf{r}}_1(x_1,t) = \left(\dot{u}_1 - \dot{\theta}_1 v_1\right)\mathbf{i}_1 + \left(\dot{v}_1 + \dot{\theta}_1 x_1 + \dot{\theta}_1 u_1\right)\mathbf{j}_1 \tag{11}$$

$$\dot{\mathbf{r}}_2(x_2,t) = \dot{\mathbf{r}}_1(L_1) + \left(\dot{u}_2 - \dot{\theta}_2 v_2\right)\mathbf{i}_2 + \left(\dot{v}_2 + \dot{\theta}_2 x_2 + \dot{\theta}_2 u_2\right)\mathbf{j}_2 \tag{12}$$

or

$$\begin{aligned} \dot{\mathbf{r}}_2(x_2,t) = (\dot{u}_{1L})\mathbf{i}_1 + (L_1 + u_{1L})\mathbf{j}_1 \\ + \left(\dot{u}_2 - \dot{\theta}_2 v_2\right)\mathbf{i}_2 + \left(\dot{v}_2 + \dot{\theta}_2 x_2 + \dot{\theta}_2 u_2\right)\mathbf{j}_2 \end{aligned} \tag{13}$$

where indices 1 and 2 correspond to the first and second links. $\dot{\theta}_1(t)$ and $\dot{\theta}_2(t)$ are the angular velocities of the coordinate frames x_1y_1 and x_2y_2, and $u_{1L} = u_1(L_1,t)$. The two coordinate systems are related as

$$\begin{aligned} \mathbf{i}_1 &= \cos(\theta_2 - \theta_1)\mathbf{i}_2 - \sin(\theta_2 - \theta_1)\mathbf{j}_2 \\ \mathbf{j}_1 &= \sin(\theta_2 - \theta_1)\mathbf{i}_2 + \cos(\theta_2 - \theta_1)\mathbf{j}_2 \end{aligned} \tag{14}$$

The kinetic energy of the links are

$$T_k = \frac{1}{2}\int_0^{L_k} \mu(x_k)(\dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k)dx_k, \quad k = 1,2 \tag{15}$$

After introducing Eqs. (11)–(14) into expression for kinetic energy and neglecting higher order terms we get expressions for the kinetic energy of the first and second links as

$$T_1 = \frac{1}{2}\int_0^{L_1} \mu(x_1)\{\dot{\theta}_1^2 v_1^2 + \dot{v}_1^2 + \dot{\theta}_1^2 x_1^2 + 2\dot{v}_1\dot{\theta}_1 x_1 + 2\dot{\theta}_1^2 x_1 u_1\}dx_1 \tag{16}$$

$$\begin{aligned} T_2 = \frac{1}{2}\int_0^{L_2} \mu(x_2)\{\dot{\theta}_1 L_1^2 + 2\dot{\theta}_1\dot{\theta}_2 L_2 x_2 \cos(\theta_2 - \theta_1) \\ -2\dot{u}_{1L}\dot{\theta}_2 x_2 \sin(\theta_2 - \theta_1) + 2\dot{\theta}_1\dot{\theta}_2 u_{1L} x_2 \cos(\theta_2 - \theta_1) \\ +2\dot{\theta}_1 L_1\left(\dot{u}_2 - \dot{\theta}_2\dot{v}_2\right)\sin(\theta_2 - \theta_1) + 2\dot{\theta}_1 L_1\left(\dot{v}_2 + \dot{\theta}_2 u_2\right)\cos(\theta_2 - \theta_1) \\ +\dot{\theta}_2^2 v_2^2 + \dot{v}_2^2 + \dot{\theta}_2^2 x_2^2 + 2\dot{v}_2\dot{\theta}_2 x_2 + 2\dot{\theta}_2^2 x_2 u_2\}dx_2 \end{aligned} \tag{17}$$

One can use simple sine functions, such as $\phi_{ki}(x) = \sin\frac{i\pi x_k}{L_k}$, $i = 1,2,...$ ($0 \leq x_k \leq L_k$ $k = 1,2$), to expend the secondary motion of each link, which can be written as

$$v_k = \sum_{i=1}^{n_k} \phi_{ki}(x_k) q_{ki}(t), \qquad k = 1,2 \tag{18}$$

We assume that the axial stretch is negligible so that the axial deformation, also known as the *shortening of the projection*, reduces to [20]

$$u_k(x_k) = -\frac{1}{2}\int_0^{x_k} \left(\frac{\partial v_k}{\partial \sigma_k}\right)^2 d\sigma_k = -\frac{1}{2}\sum_{r=1}^{n_k}\sum_{s=1}^{n_k} \mathbf{U}_{rs}^k(x_k) q_{kr}(t) q_{ks}(t) \tag{19}$$

where

$$\mathbf{U}_{rs}^k(x_k) = \int_0^{x_k} \frac{\partial \phi_{kr}(\sigma_k)}{\partial \sigma_k} \cdot \frac{\partial \phi_{ks}(\sigma_k)}{\partial \sigma_k} d\sigma_k \tag{20}$$

Using Eqs. (18)—(20) the kinetic energies can be expended as

$$T_1 = \frac{1}{2} I_{1o}\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_1^2 \underline{q}_1^T(\mathbf{m}_1 - \mathbf{h}_1)\underline{q}_1 + \frac{1}{2}\underline{\dot{q}}_1^T \mathbf{m}_1 \underline{\dot{q}}_1 + \underline{a}_1^T \underline{\dot{q}}_1 \dot{\theta}_1 \tag{21}$$

$$\begin{aligned} T_2 = {} & \frac{1}{2} I_{2o}\dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{\theta}_1^2 L_1^2 + \frac{1}{2} m_2 \dot{\theta}_1 \dot{\theta}_2 L_1 L_2 \cos(\theta_2 - \theta_1) \\ & + \frac{1}{2}\dot{\theta}_2^2 \underline{q}_2^T(\mathbf{m}_2 - \mathbf{h}_2)\underline{q}_2 + \frac{1}{2}\underline{\dot{q}}_2^T \mathbf{m}_2 \underline{\dot{q}}_2 + \underline{a}_2^T \underline{\dot{q}}_2 \dot{\theta}_2 \\ & + m_2 \dot{\theta}_1 L_1 \underline{p}_2^T \underline{\dot{q}}_2 \cos(\theta_2 - \theta_1) - m_2 \dot{\theta}_1 \dot{\theta}_2 L_1 \underline{p}_2^T \underline{q}_2 \sin(\theta_2 - \theta_1) \\ & - \frac{1}{4} m_2 \dot{\theta}_1 \dot{\theta}_2 L_2 \underline{q}_1^T \mathbf{U}_{1L} \underline{q}_1 \cos(\theta_2 - \theta_1) - \frac{1}{2} m_2 \dot{\theta}_1^2 L_1 \underline{q}_1^T \mathbf{U}_{1L} \underline{q}_1 + \frac{1}{2} m_2 \dot{\theta}_2 L2 \underline{q}_1^T \mathbf{U}_{1L} \underline{\dot{q}}_1 \\ & - m_2 \dot{\theta}_1 L_1 \underline{q}_2^T \mathbf{N}_2 \underline{\dot{q}}_2 \sin(\theta_2 - \theta_1) - \frac{1}{2} m_2 \dot{\theta}_1 \dot{\theta}_2 L_1 \underline{q}_2^T \mathbf{N}_2 \underline{q}_2 \cos(\theta_2 - \theta_1) \end{aligned} \tag{22}$$

where

$$\begin{gathered} \mathbf{m}_{kij} = \int_0^{L_k} \phi_{ki}(x_k)\phi_{kj}(x_k) dm_k, \quad \mathbf{h}_{kij} = \int_0^{L_k} \frac{1}{2}\left(L_k^2 - x_k^2\right)\left(\frac{\partial \phi_{ki}(x_k)}{\partial x_k}\right)^2 dm_k, \\ a_{ki} = \int_0^{L_k} x_k \phi_{ki}(x_k) dm_k, \quad p_{2i} = \int_0^{L_2} \phi_{2i}(x_2) dm_2, \;\; \mathbf{N}_2 = \frac{1}{L_2}\int_0^{L_2} \mathbf{U}_2(x_2) dx_2 \\ \mathbf{U}_{1L} = \mathbf{U}_1(x_1 = L_1), \qquad k = 1, 2 \end{gathered} \tag{23}$$

Expressions for kinetic energies can also be rewritten as

$$T_1 = T_1^R + T_1^v + T_1^u \tag{24}$$

$$T_2 = T_2^R + T_2^v + T_2^u + T_2^{L_1 u_2} + T_2^{L_1 v_2} + T_2^{L_2 u_1} \tag{25}$$

where

$$T_2^R = \frac{1}{2} I_{2o}\dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{\theta}_1^2 L_1^2 + \frac{1}{2} m_2 \dot{\theta}_1 \dot{\theta}_2 L_1 L_2 \cos(\theta_2 - \theta_1) \tag{26}$$

$$T_1^R = \frac{1}{2} I_{1o}\dot{\theta}_1^2 \tag{27}$$

represent kinetic energy due to the primary motion. These two terms also represent kinetic energy for rigid body model when flexibility is neglected. Existence of the secondary motion contributes to the kinetic energy as

$$T_k^v = \frac{1}{2}\dot{\theta}_k^2 \underline{q}_k^T \mathbf{m}_k \underline{q}_k + \frac{1}{2}\underline{\dot{q}}_k^T \mathbf{m}_k \underline{\dot{q}}_k + \underline{a}_k^T \underline{\dot{q}}_k \dot{\theta}_k, \quad k = 1, 2 \tag{28}$$

The shortening of the projection u has the following contribution to the kinetic energy

$$T_k^u = \frac{1}{2}\dot{\theta}_k^2 \underline{q}_k^T \mathbf{h}_k \underline{q}_k, \quad k = 1, 2 \tag{29}$$

and influence of the shortening of the projection u_1 of the first link on the kinetic energy for the second link is

$$T_2^{L_2 u_1} = -\frac{1}{4} m_2 \dot{\theta}_1 \dot{\theta}_2 L_2 \underline{q}_1^T \mathbf{U}_{1L} \underline{q}_1 \cos(\theta_2 - \theta_1) - \frac{1}{2} m_2 \dot{\theta}_1^2 L_1 \underline{q}_1^T \mathbf{U}_{1L} \underline{q}_1 + \frac{1}{2} m_2 \dot{\theta}_2 L2 \underline{q}_1^T \mathbf{U}_{1L} \underline{\dot{q}}_1 \tag{30}$$

Since the origin of the coordinate system $x_2 y_2$ is not fixed and the flexibility is included then there are extra terms in the expression for the kinetic energy of the second link

$$T_2^{L_1 u_2} = -m_2 \dot{\theta}_1 L_1 \underline{q}_2^T \mathbf{N}_2 \underline{\dot{q}}_2 \sin(\theta_2 - \theta_1) - \frac{1}{2} m_2 \dot{\theta}_1 \dot{\theta}_2 L_1 \underline{q}_2^T \mathbf{N}_2 \underline{q}_2 \cos(\theta_2 - \theta_1) \tag{31}$$

$$T_2^{L_1 v_2} = -m_2 \dot{\theta}_1 \dot{\theta}_2 L_1 \underline{p}_2^T \underline{q}_2 \sin(\theta_2 - \theta_1) + m_2 \dot{\theta}_1 L_1 \underline{p}_2^T \underline{\dot{q}}_2 \cos(\theta_2 - \theta_1) \tag{32}$$

When the axial stretch and shortening of the projection are neglected the kinetic energies for the first and second links simplify to

$$T_1 = T_1^R + T_1^v \tag{33}$$

$$T_2 = T_2^R + T_2^v + T_2^{L_1 v_2} \tag{34}$$

The Differential-Algebraic Formulation

When writing the equations of motion of a system, one first determines the number of degrees of freedom and then selects a set of generalized coordinates and, depending on the method used, generalized velocities or speeds. At this stage, one can opt for a set of independent coordinates or a set of dependent coordinates. When the constraints acting on the system are holonomic, it is possible, at least in theory, to find a set of independent generalized coordinates. When the constraints are nonholonomic equality constraints, one can find a set of independent generalized speeds.

When a set of independent coordinates are used, the number of equations of motion are the same as the number of degrees of freedom. Using a set of dependent coordinates, we introduce the Lagrange multiplier formulation to the problem and the resulting equations are in terms of the Lagrange multipliers. Denoting the Lagrange multipliers by $\lambda_j, (j = 1, ..., m)$, where m is the number of constraints, the resulting equations have the form

$$[M(\underline{w})]\underline{\ddot{w}} + \underline{h}(\underline{\dot{w}}, \underline{w}) + [A]^T \underline{\lambda} = \underline{F} \tag{35}$$

in which $[M]$ is a matrix of order $n \times n$, with n denoting the number of generalized coordinates, $[A]$ is the constraint matrix of order $m \times n$, $\underline{w}$ is generalized coordinate vector, and $\underline{\lambda}$ contains the Lagrange multipliers. The constraints can be written in the matrix form as

$$[A]\underline{\dot{w}} + \underline{b} = \underline{0} \tag{36}$$

Once the equations of motion are derived in constrained form, one can eliminate the Lagrange multipliers by means of algebraic operations. Many times such a procedure is tedious and may be harder than writing the equations of motion directly in terms of independent coordinates.

There are several complex systems where it is difficult to either write the equations of motion directly in unconstrained form, or to eliminate the Lagrange multipliers from the constrained formulation. Large mechanisms are one example. In such cases, it may be worthwhile to analyze and simulate the behavior of the system under consideration using alternate procedures that do not deal with the elimination of the Lagrange multipliers.

Differentiating Eq. (36) with respect to time

$$[A]\underline{\ddot{w}} = g^* \tag{37}$$

and combining with Eq. (35) we obtain the describing equations of the dynamical system. These equations are said to be in *differential-algebraic* form. The equations are differential with respect to the generalized coordinates, but algebraic with respect to the Lagrange multipliers, as the derivatives of the Lagrange multipliers do not appear in the formulation.

Over the past 20 years, interest in simulating and designing control laws for systems described by differential-algebraic equations has seen tremendous interest [12], [15–19], [21–22]. There now exists powerful software to simulate systems described by differential-algebraic equations [12], [17]. We take advantage of these developments and introduce the differential-algebraic formulation to mechanisms that have elastic arms.

Elastic Closed Kinematic Chains

We will illustrate the modeling procedure by a simple-looking example. Consider the four bar mechanism in Fig. 3. When the links are rigid, the mechanism has one degree of freedom. Writing the equation of motion in unconstrained form using one generalized coordinate, say θ_1, proves to be complicated [16], so we explore writing the equations of motion in constrained form.

We use θ_1, θ_2, and θ_3 as the generalized coordinates and treat links 1 and 2 as one system and link 3 as another. As a result, we have three equations of motion and two constraint equations, corresponding to the displacement of point P being common to both systems. An alternate method of writing the equations would be to treat each link separately and use five generalized coordinates and four constraints [16].

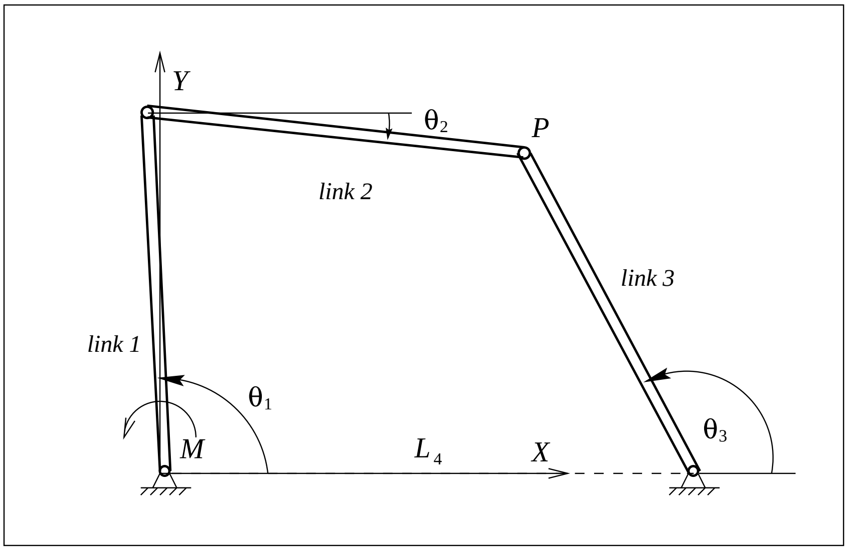

Figure 3: The Four Bar Mechanism

We consider motion in a horizontal plane and we are ignoring elastic effects for the time being, so that the potential energy is zero. The Lagrangian has the form

$$L = T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 + \frac{1}{2}m_2 v_G^2 \tag{38}$$

where I_1, and I_3 are mass moments of inertia of the first and third links about their fixed pin joints and I_2 is the mass moment of inertia of the second link about its center of mass. We denote by m_2 and v_G the mass and velocity of the center of mass of the second link. The velocity of the center of mass of the second link has the form

$$v_G = -\Big(L_1\dot{\theta}_1 \sin\theta_1 + L_2\dot{\theta}_2 \sin\theta_2\Big)\mathbf{I} + \Big(L_1\dot{\theta}_1 \cos\theta_1 + L_2\dot{\theta}_2 \cos\theta_2\Big)\mathbf{J} \tag{39}$$

where the unit vectors associated with the inertial frame XY are denoted by $\mathbf{I}$ and $\mathbf{J}$. The virtual work has the form

$$\delta W = \mathcal{M}\delta\theta_1 \tag{40}$$

and the constaints are

$$\begin{gathered} L_1 \cos\theta_1 + L_2 \cos\theta_2 - L_3 \cos\theta_3 - L_4 = 0 \\ L_1 \sin\theta_1 + L_2 \sin\theta_2 - L_3 \sin\theta_3 = 0 \end{gathered} \tag{41}$$

We obtain the equations of motion as

$$\begin{gathered} \big(I_1 + m_2 L_1^2\big)\ddot{\theta}_1 + m_2 L_1 L_2 \cos(\theta_2 - \theta_1)\ddot{\theta}_2 - m_2 L_1 L_2 \sin(\theta_2 - \theta_1)\dot{\theta}_2 + \\ \lambda_1 L_1 \sin\theta_1 - \lambda_2 L_1 \cos\theta_1 = \mathcal{M} \\ \big(I_2 + m_2 L_2^2\big)\ddot{\theta}_2 + m_2 L_1 L_2 \cos(\theta_2 - \theta_1)\ddot{\theta}_1 + m_2 L_1 L_2 \sin(\theta_2 - \theta_1)\dot{\theta}_1 + \\ \lambda_1 L_2 \sin\theta_2 - \lambda_2 L_2 \cos\theta_2 = 0 \\ I_3\ddot{\theta}_3 + \lambda_1 L_3 \sin\theta_3 + \lambda_2 L_3 \cos\theta_3 = 0 \end{gathered} \tag{42}$$

The complexity of these equations should be compared with the equation of motion in terms of the single unconstrained coordinate, given in [16]. Considering that we are about to introduce flexibility effects, the complexity of the equations that would be resulting using unconstrained coordinates will be much worse. Hence, we dispense with the unconstrained formulation and use the zero tip deformation constraint to describe the secondary motion.

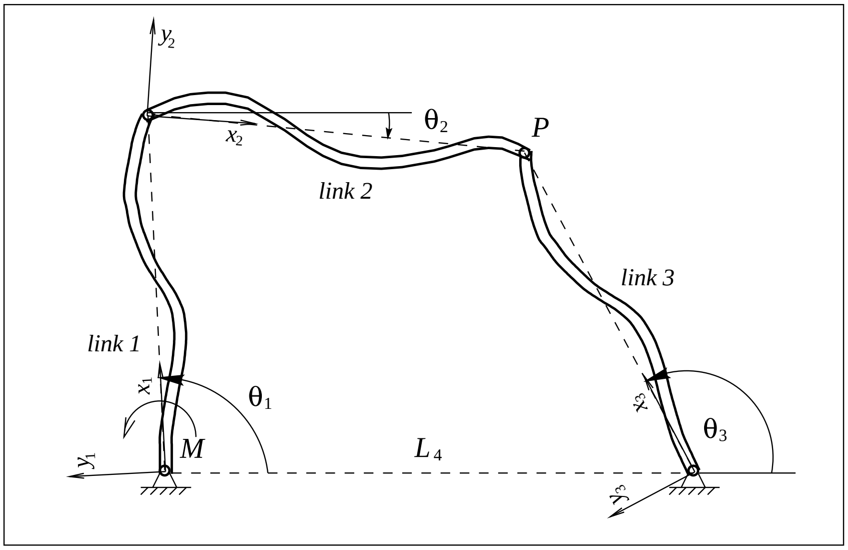

Figure 4: Flexible Four Bar Mechanism

Let us only consider deformation in the plane. Figure 4 shows the selection of the reference frames. We attach a moving frame x_1y_1 to the first link, a frame x_2y_2 to the second link, and a frame x_3y_3 to the third link. The angular velocities of the reference frames are $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$, respectively. The kinetic energy expressions for links 1 and 2 are given in Eqs. (24) and (25). The kinetic energy for the third link has the same form as the kinetic energy of the first link:

$$\begin{aligned} T_1 &= T_1^R + T_1^v + T_1^u \\ &= \frac{1}{2}I_{1o}\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_1^2\underline{q}_1^T(\mathbf{m}_1 - \mathbf{h}_1)\underline{q}_1 + \frac{1}{2}\dot{\underline{q}}_1^T\mathbf{m}_1\dot{\underline{q}}_1 + \underline{a}_1^T\dot{\underline{q}}_1\dot{\theta}_1 \end{aligned} \tag{43}$$

$$\begin{aligned} T_2 &= T_2^R + T_2^v + T_2^u + T_2^{L_1u_2} + T_2^{L_1v_2} + T_2^{L_2u_1} \\ &= \frac{1}{2}I_{2o}\dot{\theta}_2^2 + \frac{1}{2}m_2\dot{\theta}_1^2L_1^2 + \frac{1}{2}m_2\dot{\theta}_1\dot{\theta}_2L_1L_2\cos(\theta_2-\theta_1) \\ &\quad + \frac{1}{2}\dot{\theta}_2^2\underline{q}_2^T(\mathbf{m}_2 - \mathbf{h}_2)\underline{q}_2 + \frac{1}{2}\dot{\underline{q}}_2^T\mathbf{m}_2\dot{\underline{q}}_2 + \underline{a}_2^T\dot{\underline{q}}_2\dot{\theta}_2 \\ &\quad + m_2\dot{\theta}_1L_1\underline{p}_2^T\dot{\underline{q}}_2\cos(\theta_2-\theta_1) - m_2\dot{\theta}_1\dot{\theta}_2L_1\underline{p}_2^T\underline{q}_2\sin(\theta_2-\theta_1) \\ &\quad - \frac{1}{4}m_2\dot{\theta}_1\dot{\theta}_2L_2\underline{q}_1^T\mathbf{U}_{1L}\underline{q}_1\cos(\theta_2-\theta_1) - \frac{1}{2}m_2\dot{\theta}_1^2L_1\underline{q}_1^T\mathbf{U}_{1L}\underline{q}_1 + \frac{1}{2}m_2\dot{\theta}_2L2\underline{q}_1^T\mathbf{U}_{1L}\dot{\underline{q}}_1 \\ &\quad - m_2\dot{\theta}_1L_1\underline{q}_2^T\mathbf{N}_2\dot{\underline{q}}_2\sin(\theta_2-\theta_1) - \frac{1}{2}m_2\dot{\theta}_1\dot{\theta}_2L_1\underline{q}_2^T\mathbf{N}_2\underline{q}_2\cos(\theta_2-\theta_1) \end{aligned} \tag{44}$$

$$
\begin{aligned}
T_3 &= T_3^R + T_3^v + T_3^u \\
&= \frac{1}{2} I_{3o} \dot{\theta}_3^2 + \frac{1}{2} \dot{\theta}_3^2 \underline{q}_3^T (\mathbf{m}_3 - \mathbf{h}_3) \underline{q}_3 + \frac{1}{2} \underline{\dot{q}}_3^T \mathbf{m}_3 \underline{\dot{q}}_3 + \underline{a}_3^T \underline{\dot{q}}_3 \dot{\theta}_3
\end{aligned}
\tag{45}
$$

where I_{1o}, I_{2o}, and I_{3o} are mass moments of inertia of links 1 ,2 and 3 with respect to the origins on corresponding moving reference frames. The even numbered terms of $a_{ki}(k = 1, 2, 3\ ,\ \ i = 2, 4, ...)$ disappear as a result of the symmetry properties of the sine function. Also, because of the properties of the sine function, $\mathbf{m}_k$ ($k = 1, 2, 3$) are diagonal matrices, where the diagonal terms depend on the mass properties. As a result of the elasticity of the arm, we have potential energy associated with the system, which can be expressed as

$$
V_k(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} K_{kij}, \quad k = 1, 2, 3 \tag{46}
$$

where

$$
K_{kij} = \int_0^{L_k} EI(x_k) \frac{\partial^2 \phi_{ki}(x_k)}{\partial x_k^2} \frac{\partial^2 \phi_{kj}(x_k)}{\partial x_k^2} dx_k, \quad i, j = 1, 2, ..., n \tag{47}
$$

When the arm is uniform, K_{kij} simplifies to $\Omega_{ki} \mathbf{m}_{kii} \underline{\delta}_{ij}$ in which Ω_{ki} are the natural frequencies of the *k*-th arm. The equations of motion can now be obtained by taking the appropriate derivatives of the kinetic and potential energies.

The Lagrangian has the form

$$
\begin{aligned}
L &= T_1 + T_2 + T_3 - V_1 - V_2 - V_3 \\
&= T_1^R + T_1^v + T_1^u + T_2^R + T_2^v + T_2^u + T_2^{L_1 u_2} + T_2^{L_1 v_2} + T_2^{L_2 u_1} \\
&\quad + T_3^R + T_3^v + T_3^u - V_1 - V_2 - V_3
\end{aligned}
\tag{48}
$$

The virtual work has the form

$$
\delta W = \mathcal{M} \delta \theta_1 + \mathcal{M} \underline{\phi}'(0) \delta \underline{q} \tag{49}
$$

The constraint equations are

$$
\begin{aligned}
(L_1 + u_1(L_1, t)) \cos\theta_1 + (L_2 + u_2(L_2, t)) \cos\theta_2 &= (L_3 + u_3(L_3, t)) \cos\theta_3 + b \\
(L_1 + u_1(L_1, t)) \sin\theta_1 + (L_2 + u_2(L_2, t)) \sin\theta_2 &= (L_3 + u_3(L_3, t)) \sin\theta_3
\end{aligned}
\tag{50}
$$

The equations of motion can be shown to be

$$
\begin{aligned}
[M(\underline{w})] \underline{\ddot{w}} + \underline{h}(\underline{\dot{w}}, \underline{w}) + [A]^T \underline{\lambda} &= \underline{F} \\
[A] \underline{\ddot{w}} &= g^*
\end{aligned}
\tag{51}
$$

where

$$
[M] = \begin{bmatrix} \left([M]_{123}^R + [M]_{123}^{Ru} + [M]_{123}^{Rv} \right) & \left([M]_{123}^{qu} + [M]_{123}^{qv} \right) \\ \left([M]_{123}^{qu} + [M]_{123}^{qv} \right)^T & [M]_q \end{bmatrix} \tag{52}
$$

$$
\underline{h} = \begin{bmatrix} h_1^R + h_1^{u_1} + h_1^{v_1} + h_1^{u_2} + h_1^{v_2} \\ h_2^R + h_2^{u_1} + h_2^{u_2} + h_2^{v_2} \\ h_3^{u_3} + h_3^{v_3} \\ h_{q_1}^{u_1} + h_{q_1}^{v_1} \\ h_{q_2}^{u_2} + h_{q_2}^{v_2} \\ h_{q_3}^{u_3} + h_{q_3}^{v_3} \end{bmatrix} \tag{53}
$$

$$
[A] = \left[\left([A]_{123}^R + [A]_{123}^u \right) \quad [A]_{q_1} \quad [A]_{q_2} \quad [A]_{q_3} \right] \tag{54}
$$

$$
g^* = g^R + g^{u_1} + g^{u_2} - g^{u_3} \tag{55}
$$

and $\underline{F}$ is a vector of length $n = 1 + n_1 + n_2 + n_3$ and n_k ($k = 1, 2, 3$) is the number of trial functions used for each link. Expressions for the submatrices in the above equations are given in the appendix.

When the shortening of the projection is not considered, then equation of motion is the same as Eq. (51), but the matrices inside the equation are given as

$$
[M] = \begin{bmatrix} \left([M]_{123}^R + [M]_{123}^{Rv} \right) & [M]_{123}^{qv} \\ \left([M]_{123}^{qv} \right)^T & [M]_q \end{bmatrix}, \quad \underline{h} = \begin{bmatrix} h_1^R + h_1^{v_1} + h_1^{v_2} \\ h_2^R + h_2^{v_2} \\ h_3^{v_3} \\ h_{q_1}^{v_1} \\ h_{q_2}^{v_2} \\ h_{q_3}^{v_3} \end{bmatrix} \tag{56}
$$

$$
[A] = [A]_{123}^R, \quad g^* = g^R \tag{57}
$$

and $\underline{F}$ is a vector of length $n = 1 + n_1 + n_2 + n_3$.

When only rigid body model is considered, then the equation of motion is also given by Eq. (51) with the following matrices

$$
[M] = [M]_{123}^R, \quad \underline{h} = \begin{bmatrix} h_1^R \\ h_2^R \\ 0 \end{bmatrix} \tag{58}
$$

$$
[A] = [A]_{123}^R, \quad g^* = g^R \tag{59}
$$

and $\underline{F}$ is a vector of length $n = 1$.

Analysis of the Equations of Motion

The equations of motion for both the rigid body and flexible models have the same form. The difference between the two can be seen by comparing the expressions for matrices that are used in the equation of motion. They are given by Eqs. (51),(58),(59) for the rigid body model and by Eqs. (51),(53)-(55) or Eqs. (51),(56),(57) for the flexible models. Deriving the equation of motion for the rigid body model in terms of independent generalized coordinates is difficult and introducing flexibility to the model adds extra complexity. Hence, the differential-algebraic formulation becomes a viable approach.

Another question is whether it is necessary to include shortening of the projection into the model. The answer to that question can be found by comparing the equations of motion. It is obvious that equation of motion that include the shortening are more complex than ones without shortening. To determine how much the shortening of the projection is important we simulate the motion by using both models. We consider that the mechanism moves on a horizontal plane (i.e., there is no gravitational potential energy), and there is no external forcing (i.e., $\underline{F} = \underline{0}$). The total system energy is conserved (i.e., $E = T + V = constant$). For all simulations it is assumed that links are made of steel (density is $\rho = 7830kg/m^3$ and elastic modulus is $E = 20.7GPa$). The cross section of the links are $10mm \times 8mm$ rectangles. The lengths of the links are $L_1 = 1m$, $L_2 = 1.2m$, $L_3 = 1m$, and $L_4 = 1.2m$. We choose as independent coordinates θ_1, $\underline{q}_1$, $\underline{q}_2$, and $\underline{q}_3$, while θ_2 and θ_3 are dependent coordinate. Initial values for the independent coordinates are $\theta_1 = \pi/3rad$, $\dot{\theta}_1 = 1.5rad/sec$, $\underline{q}_1 = 10^{-3} \times [5 \quad 0.1\,]^T m$, $\underline{q}_2 = 10^{-3} \times [5 \quad 0.1\,]^T m$, $\underline{q}_3 = 10^{-3} \times [50 \quad 0.1\,]^T m$, $\dot{\underline{q}}_1 = [0 \quad 0\,]^T m/s$, $\dot{\underline{q}}_2 = [0 \quad 0\,]^T m/s$, and $\dot{\underline{q}}_3 = [0 \quad 0\,]^T m/s$. The natural frequencies for the first and third likes are $\Omega_{11} = \Omega_{31} = 117.2s^{-1}$ and $\Omega_{12} = \Omega_{32} = 468.8s^{-1}$. The second link has the natural frequencies $\Omega_{21} = 81.38s^{-1}$ and $\Omega_{22} = 325.5s^{-1}$.

We simulate differential-algebraic equations (DAE) as follows: first we decide which coordinates are the independent ones and then we eliminate Lagrange multiplier $\underline{\lambda}$ and acceleration terms of dependent coordinates (in our case $\ddot{\theta}_2$ and $\ddot{\theta}_3$) from Eq. (51). That new differential equation contains only acceleration terms of independent coordinates, but it is a function of displacement and velocity of both independent and dependent coordinates. By using given initial values of the independent coordinates we solve constraint equations for θ_2, $\dot{\theta}_2$, θ_3, and $\dot{\theta}_3$. These values are used to numerically solve new differential equation for independent coordinates. This whole cycle is repeated for each interval of time. After each cycle we have to check whether independent coordinates are chosen properly so that constraint equations can be solved for the dependent coordinates. If that is not possible a new set of independent coordinates has to chosen [13]. Note that DAE can also be solved by using available softwares, such as DASSL [12].

Figure 5 shows the total system energy for models with and without shortening. The total energy is little bit higher when the shortening is included.

In order to analyze how much the shortening contributes to the total energy we first simulate the motion by using the model with shortening. The total energy of the system in that case is

$$\begin{aligned} E = T_1^R + T_1^v + T_1^u + T_2^R + T_2^v + T_2^u + T_2^{L_1 u_2} + T_2^{L_1 v_2} + T_2^{L_2 u_1} \\ + T_3^R + T_3^v + T_3^u + V_1 + V_2 + V_3 \end{aligned} \tag{60}$$

Using the system response from that simulation we calculate the total energy without terms that are caused by the shortening or

$$\begin{aligned} E_c = T_1^R + T_1^v + T_2^R + T_2^v + T_2^{L_1 v_2} \\ + T_3^R + T_3^v + V_1 + V_2 + V_3 \end{aligned} \tag{61}$$

This is the same as the total energy of the system for the model without shortening. The values of E_c are presented in Fig. 6. These results correspond to measuring the real system response and calculating the total energy by using Eq. (61). It is clear that the shortening of the projection is not negligible.

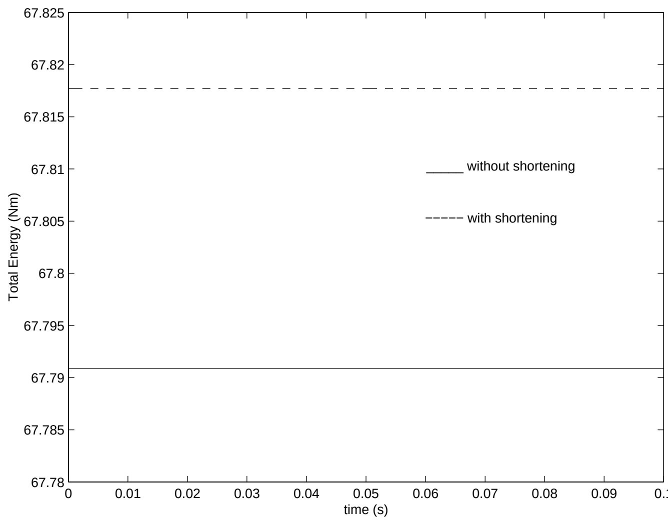

Figure 5: Total Energy of the system

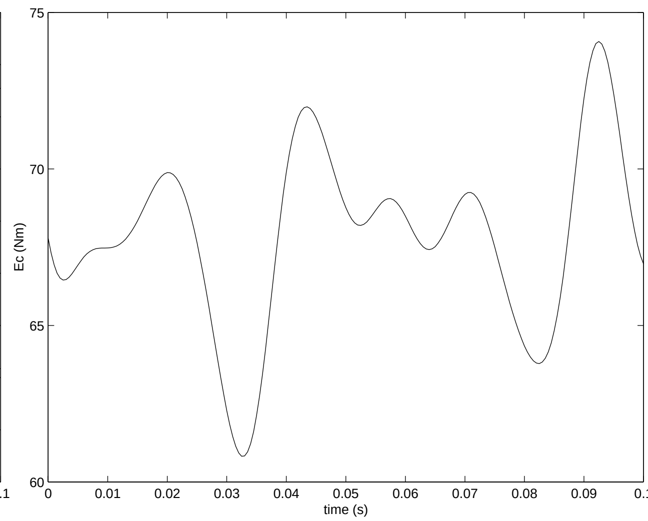

Figure 6: E_c

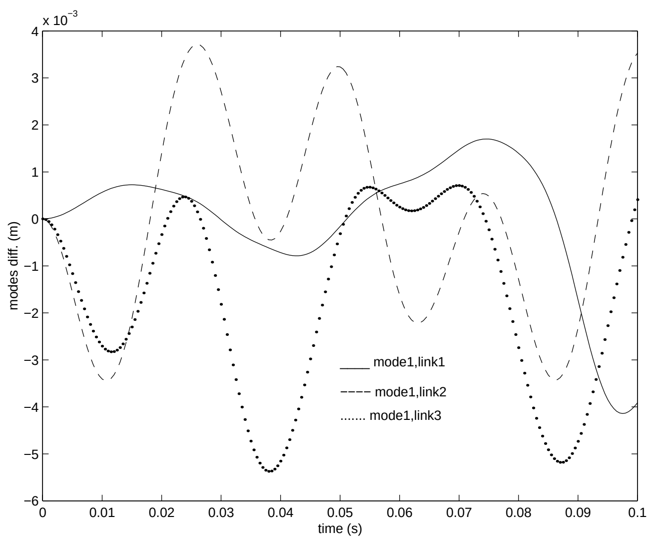

Figure 7: Difference between The First Modes

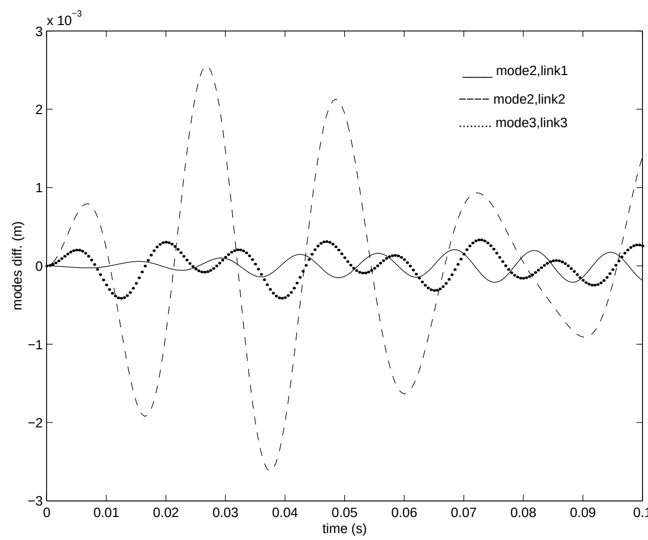

Figure 8: Difference between The Second Modes

As long as we use the same model (either with or without shortening) to simulate the motion and calculate the total energy, from Fig. 5 we see that the total energy of the system is constant. The problem arises when we use the real system to measure the response but we design a control law or perform some other analyses based on the model without shortening.

Note that shortening of the projection does not play an important role when there is only one link. By contrary, when modeling multilink flexible mechanisms shortening of the projection can not be neglected. Figures 7 and 8 show the difference between the response of the first two modes, for the models with and without shortening.

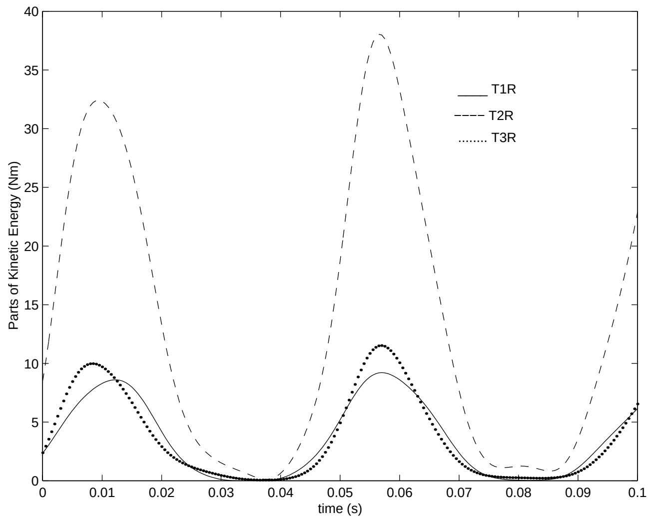

Figure 9: T_1^R, T_2^R, and T_3^R Parts of The Kinetic Energy

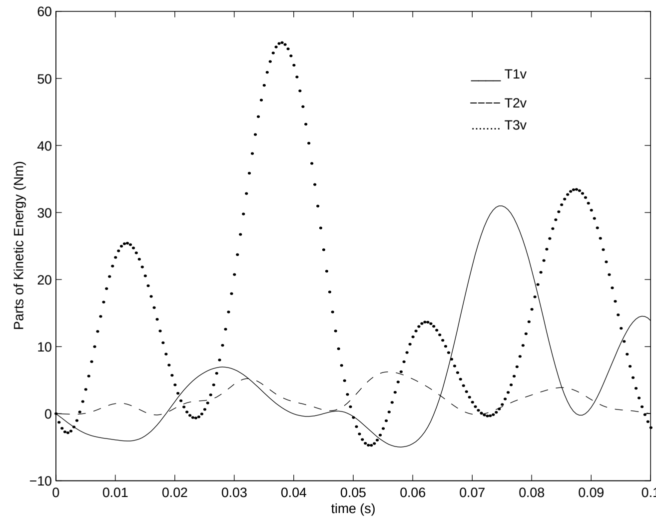

Figure 10: T_1^v, T_2^v, and T_3^v Parts of The Kinetic Energy

Not all the terms in Eq. (60) are of the same magnitude. Figures 9–13 show the time history for different parts of the kinetic energy. Comparing Fig. 11 with other figures we see that T_1^u, T_2^u, and T_3^u parts of kinetic energy are the smallest. On the contrary, $T_2^{L_1 u_2}$ and $T_2^{L_2 u_1}$ are much larger. These two parts of the kinetic energy exist because the second link is attached to the first link and the origin of the coordinate system $x_2 y_2$ is moving. From Figs. 9–13 we see that the shortening of the projection should not be neglected when there is a multilink mechanism.

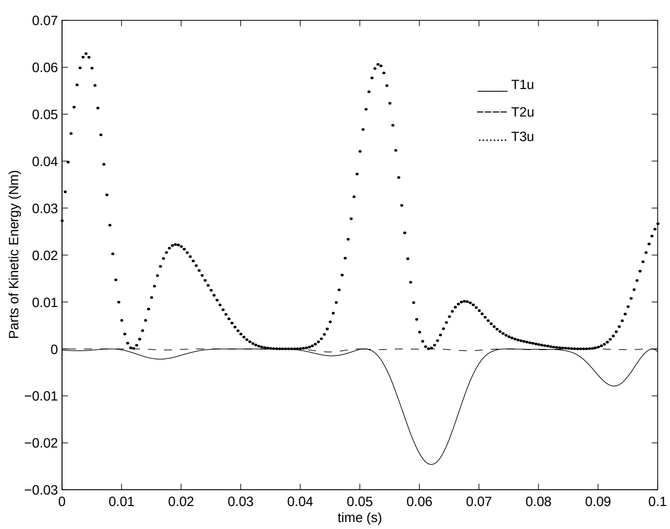

Figure 11: T_1^u, T_2^u, and T_3^u parts of The Kinetic Energy

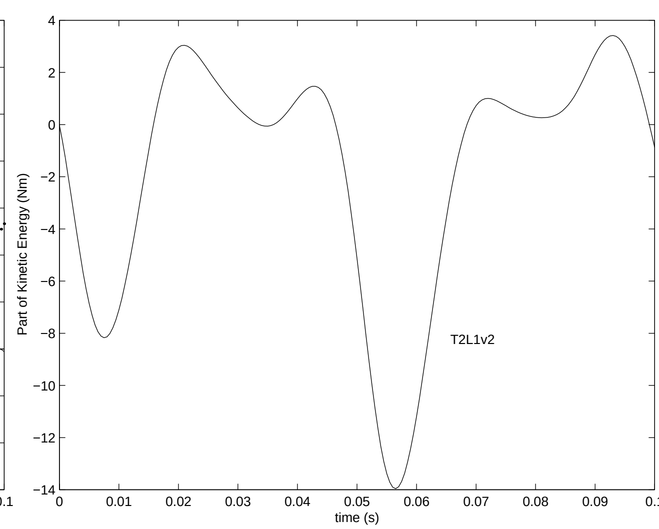

Figure 12: $T_2^{L_1 v_2}$ part of The Kinetic Energy

From the previous discussion it may appear that the shortening u_3 can be neglected completely. That is not correct because u_1, u_2, and u_3 influence all of the generalized coordinates through the constraint equations. We demonstrate this by performing the simulation by using the model with the shortening (ie., both dependent and independent coordinates are obtained by using the shortening). The total energy is calculated by using simulation results for independent coordinates but the dependent coordinates θ_2 and θ_3 are obtained from Eq. (59) (ie., shortening of the projection is not included in the constraint equations). That case is presented in Figure 14. As expected the total energy is not constant.

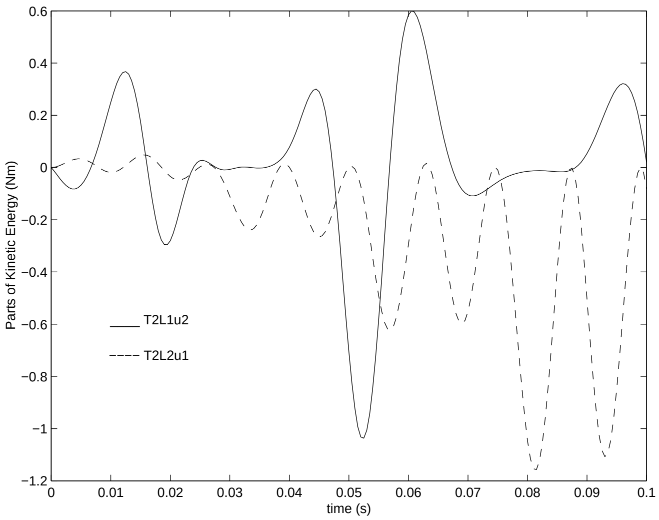

Figure 13: $T_2^{L_1 u_2}$ and $T_2^{L_2 u_1}$ parts of The Kinetic Energy

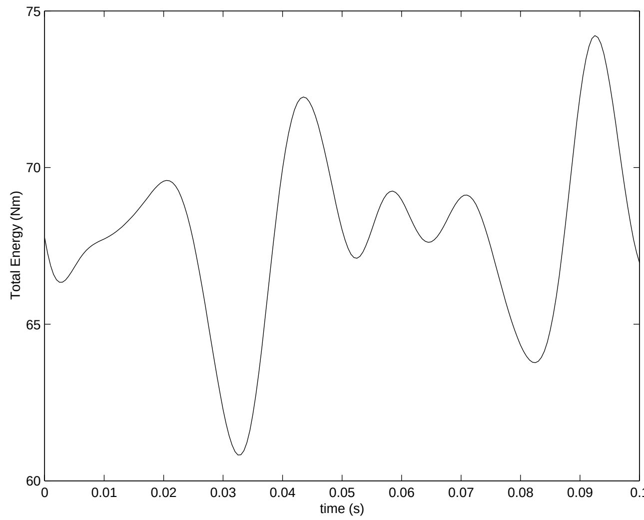

Figure 14: The Total energy

Conclusions

In this paper we present a new way of modeling complex mechanical systems that are comprised of elastic components, particularly closed chains. We view the motion of each link using a moving reference frame and introduce the zero tip deformation constraint. We combine this analysis with the differential-algebraic formulation and write the equation of motion in constrained form. We analyze the form and solution of the resulting equations.

We also analyze whether it is necessary to include the shortening of the projection when the links are flexible. We conclude that shortening can not be neglected when there are two or more interconnected links.

Appendix

Here, we provide expressions for submatrices and other parameters that are used in the equations of motion given by Eqs. (51)-(59).

$$[M]_{123}^{R} = \begin{bmatrix} I_{1o} + m_2 L_1^2 & 0.5 m_2 L_1 L_2 \cos_{tt} & 0 \\ 0.5 m_2 L_1 L_2 \cos_{tt} & I_{2o} & 0 \\ 0 & 0 & I_{3o} \end{bmatrix} \tag{62}$$

$$[M]_{123}^{Rv} = \begin{bmatrix} \underline{q}_1^T [\mathbf{m}_1] \underline{q}_1 - m_2 L_2 \underline{p}_2^T \underline{q}_2 \sin_{tt} & 0 & 0 \\ 0 & \underline{q}_2^T [\mathbf{m}_2] \underline{q}_2 & 0 \\ 0 & 0 & \underline{q}_3^T [\mathbf{m}_3] \underline{q}_3 \end{bmatrix} \tag{63}$$

$$[M]^{Ru}_{123} = \begin{bmatrix} -\underline{q}_1^T[\mathbf{h}_1]\underline{q}_1 - m_2L_1\underline{q}_1^T\mathbf{U}_{1L}\underline{q}_1 & \# & 0 \\ \# & -\underline{q}_2^T[\mathbf{h}_2]\underline{q}_2 & 0 \\ 0 & 0 & -\underline{q}_3^T[\mathbf{h}_3]\underline{q}_3 \end{bmatrix} \tag{64}$$

where $\# = -m_2\Big(0.25L_2\underline{q}_1^T\mathbf{U}_{1L}\underline{q}_1 + 0.5L_1\underline{q}_2^T\mathbf{N}_2\underline{q}_2\Big)\cos_{tt}$, $\cos_{tt} = \cos(\theta_2-\theta_1)$, $\sin_{tt} = \sin(\theta_2-\theta_1)$, and $[M]_{q_k} = [\mathbf{m}_k]$, $k = 1,2,3$.

$$[M]^{qu}_{123} = \begin{bmatrix} 0 & -m_2L_1\underline{q}_2^T\mathbf{N}_2\sin_{tt} & 0 \\ 0.5m_2L_2\underline{q}_1^T\mathbf{U}_{1L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{65}$$

$$[M]^{qv}_{123} = \begin{bmatrix} \underline{a}_1^T & m_2L_1\underline{p}_2^T\cos_{tt} & 0 \\ 0 & \underline{a}_2^T & 0 \\ 0 & 0 & \underline{a}_3^T \end{bmatrix}, \quad [M]_q = \begin{bmatrix} [\mathbf{m}_1] & 0 & 0 \\ 0 & [\mathbf{m}_2] & 0 \\ 0 & 0 & [\mathbf{m}_3] \end{bmatrix} \tag{66}$$

$$h_1^R = -0.5m_2L_1L_2\dot\theta_2^2\sin_{tt}, \quad h_1^{v_1} = 2\underline{q}_1^T[\mathbf{m}_1]\underline{\dot q}_1\dot\theta_1 \tag{67}$$

$$h_1^{u_1} = 2\underline{q}_1^T[\mathbf{h}_1]\underline{\dot q}_1\dot\theta_1 - 2m_2L_1\dot\theta_1\underline{q}_1\mathbf{U}_{1L}\underline{\dot q}_1^T + 0.25m_2L_2\dot\theta_2^2\underline{q}_1\mathbf{U}_{1L}\underline{q}_1\sin_{tt} \tag{68}$$

$$h_1^{u_2} = -m_2L_1\underline{\dot q}_2^T\mathbf{N}_2\underline{\dot q}_2\sin_{tt} - 2m_2L_1\dot\theta_2\underline{q}_2^T\mathbf{N}_2\underline{\dot q}_2\cos_{tt} - 0.5m_2L_1\dot\theta_2^2\underline{q}_2^T\mathbf{N}_2\underline{q}_2\sin_{tt} \tag{69}$$

$$h_1^{v_2} = -2m_2L_1\dot\theta_2\underline{p}_2^T\underline{\dot q}_2\sin_{tt} - m_2L_1\dot\theta_2^2\underline{p}_2^T\underline{q}_2\cos_{tt} \tag{70}$$

$$h_2^R = 0.5m_2L_1L_2\dot\theta_1^2\sin_{tt} \tag{71}$$

$$h_2^{u_1} = 0.5m_2L_2\underline{\dot q}_1^T\mathbf{U}_{1L}\underline{\dot q}_1\sin_{tt} - m_2L_2\dot\theta_1\underline{q}_1^T\mathbf{U}_{1L}\underline{\dot q}_1\cos_{tt} - 0.25m_2L_2\dot\theta_1^2\underline{q}_1^T\mathbf{U}_{1L}\underline{q}_1\sin_{tt} \tag{72}$$

$$h_2^{u_2} = -2\dot\theta_2\underline{q}_2^T[\mathbf{h}_2]\underline{\dot q}_2 - 0.5m_2L_1\dot\theta_1^2\underline{q}_2^T\mathbf{N}_2\underline{q}_2\sin_{tt} \tag{73}$$

$$h_2^{v_2} = 2\dot\theta_2\underline{q}_2^T[\mathbf{m}_2]\underline{\dot q}_2 + m_2L_1\dot\theta_1^2\underline{p}_2^T\underline{q}_2\cos_{tt} \tag{74}$$

$$h_3^R = 0, \quad h_3^{u_3} = -2\dot\theta_3\underline{q}_3^T[\mathbf{h}_3]\underline{\dot q}_3, \quad h_3^{v_3} = 2\dot\theta_3\underline{q}_3^T[\mathbf{m}_3]\underline{\dot q}_3, \tag{75}$$

$$h_{q_1}^{v_1} = -\dot\theta_1^2[\mathbf{m}_1]\underline{q}_1 \tag{76}$$

$$h_{q_1}^{u_1} = \dot\theta_1^2[\mathbf{h}_1]\underline{q}_1 + 0.5m_2L_2\dot\theta_2^2\mathbf{U}_{1L}\underline{q}_1\cos_{tt} + m_2L_1\dot\theta_1^2\mathbf{U}_{1L}\underline{q}_1 \tag{77}$$

$$h_{q_2}^{u_2} = \dot\theta_2^2[\mathbf{h}_2]\underline{q}_2 + m_2L_1\dot\theta_1^2\mathbf{N}_2\underline{q}_2\cos_{tt} \tag{78}$$

$$h_{q_2}^{v_2} = -\dot\theta_2^2[\mathbf{m}_2]\underline{q}_2 + m_2L_1\dot\theta_1^2\underline{p}_2\sin_{tt} \tag{79}$$

$$h_{q_3}^{u_3} = \dot\theta_3^2[\mathbf{h}_3]\underline{q}_3, \quad h_{q_3}^{v_3} = -\dot\theta_3^2[\mathbf{m}_3]\underline{q}_3 \tag{80}$$

$$[A]^R_{123} = \begin{bmatrix} -L_1\sin\theta_1 & -L_2\sin\theta_2 & L_3\sin\theta_3 \\ L_1\cos\theta_1 & L_2\cos\theta_2 & -L_3\cos\theta_3 \end{bmatrix} \tag{81}$$

$$[A]^u_{123} = \begin{bmatrix} 0.5\underline{q}_1^T\mathbf{U}_{1L}\underline{q}_1\sin\theta_1 & 0.5\underline{q}_2^T\mathbf{U}_{2L}\underline{q}_2\sin\theta_2 & -0.5\underline{q}_3^T\mathbf{U}_{3L}\underline{q}_3\sin\theta_3 \\ -0.5\underline{q}_1^T\mathbf{U}_{1L}\underline{q}_1\cos\theta_1 & -0.5\underline{q}_2^T\mathbf{U}_{2L}\underline{q}_2\cos\theta_2 & 0.5\underline{q}_3^T\mathbf{U}_{3L}\underline{q}_3\cos\theta_3 \end{bmatrix} \tag{82}$$

$$[A]_{q_k} = \begin{bmatrix} -\underline{q}_k^T\mathbf{U}_{kL}\cos\theta_k \\ -\underline{q}_k^T\mathbf{U}_{kL}\sin\theta_k \end{bmatrix}, \quad k = 1,2,3 \tag{83}$$

$$g^R = \begin{bmatrix} -L_1\dot{\theta}_1^2\cos\theta_1 - L_2\dot{\theta}_2^2\cos\theta_2 + L_3\dot{\theta}_3^2\cos\theta_3 \\ -L_1\dot{\theta}_1^2\sin\theta_1 - L_2\dot{\theta}_2^2\sin\theta_2 + L_3\dot{\theta}_3^2\sin\theta_3 \end{bmatrix} \tag{84}$$

$$g^{u_k} = \begin{bmatrix} 0.5\dot{\theta}_k^2\underline{q}_k^T\mathbf{U}_{kL}\underline{q}_1\cos\theta_k + 2\dot{\theta}_k\underline{q}_k^T\mathbf{U}_{kL}\dot{\underline{q}}_k\sin\theta_k - \dot{\underline{q}}_k^T\mathbf{U}_{kL}\dot{\underline{q}}_k\cos\theta_k \\ 0.5\dot{\theta}_k^2\underline{q}_k^T\mathbf{U}_{kL}\underline{q}_1\sin\theta_k - 2\dot{\theta}_k\underline{q}_k^T\mathbf{U}_{kL}\dot{\underline{q}}_k\cos\theta_k - \dot{\underline{q}}_k^T\mathbf{U}_{kL}\dot{\underline{q}}_k\sin\theta_k \end{bmatrix} \quad k = 1, 2, 3 \tag{85}$$

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