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Predication of nonlinear heat transfer in a convective-radiative fin with temperature-dependent properties by the collocation spectral method

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ABSTRACT
The applicability of the collocation spectral method (CSM) for solving nonlinear heat transfer problems is demonstrated in a convective-radiative fin with temperature-dependent properties. In this method, the fin temperature distribution is approximated by Lagrange interpolation polynomials at spectral collocation points. The differential form of the energy equation is transformed to a matrix form of algebraic equations. The computational convergence of the CSM approximately follows an exponential decaying law; and thus, it is a very simple and effective approach for a rapid assessment of nonlinear physical problems. The effects of temperature-dependent properties such as thermal conductivity, surface emissivity, heat transfer coefficient, convection-conduction parameter, and radiation-conduction parameter on the fin temperature distribution and efficiency are discussed.

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1. Introduction
Rapid removal of heat generated in compact electronic components/devices is in great demand with advances of microelectronics technologies. The fin is a simple yet effective instrument used to enhance heat transfer from a heated surface to environment [1]. In traditional fin heat transfer analysis, thermal conductivity is generally assumed constant. When there exists a large temperature variation, however, the dependence of thermal conductivity on temperature must be considered. For instance, the thermal conductivity of AISI 302 stainless steel increases from 17.3 W/m K at 400 K to 25.5 W/m K at 1,000 K [2].

A fin dissipates heat to the environment via convection and radiation. The convective heat transfer coefficient is also dependent on temperature difference, and can be generally expressed as a power-law form \( h \propto (\Delta T)^n \), in which the power value \( n \) depends on the mechanism of convective heat transfer. For example, typical power values are \( n = 1/4 \) for laminar natural convection and \( n = 1/3 \) for turbulent natural convection [3]. Further, the heat generation rate can also be a function of temperature. Aziz [4] analyzed the performance of a convective fin with internal heat generation by a perturbation method. Razani and Ahmadi [5] optimized circular fins with an arbitrary heat source distribution and a nonlinear temperature-dependent thermal conductivity. Aziz and Bouaziz [6]...
adopted a least-squares method to evaluate the fin efficiency of a longitudinal fin with temperature-dependent internal heat generation and thermal conductivity. Ghasemi et al. [7] developed the differential transformation method (DTM) to solve the nonlinear temperature distribution equation in a longitudinal fin with temperature-dependent internal heat generation. Kundu and Lee [8] analyzed the temperature distribution and fin efficiency in a fin with internal heat generation for both Fourier and non-Fourier heat conduction mechanisms.

In the case of high temperature, radiation dissipation from fin surfaces plays a critical role. The emissivity of a real surface is not a constant, but rather variable with temperature [9, 10]. Torabi and Aziz [11] employed the DTM to analyze the thermal performance and efficiency of a T-shape fin of combined convection and radiation. They considered the temperature dependence of thermal conductivity, heat transfer coefficient, and surface emissivity. Arslanturk [12] used the homotopy perturbation method to evaluate temperature distribution within straight radiating fins with a step change in thickness and variable thermal conductivity. Torabi et al. [13] used the DTM to establish the performance characterization of convective-radiative longitudinal fins of rectangular, trapezoidal, and concave parabolic profiles with temperature-dependent thermal conductivity, heat transfer coefficient, and surface emissivity. Recently, Saedodin and Barforoush [14] extended the DTM to analyze the thermal processing of moving convective-radiative plates with temperature-dependent thermal conductivity.

The collocation spectral method (CSM) is a high-order numerical method that is based on Chebyshev or Fourier polynomials [15]. It can provide exponential convergence [16] to expedite the computation. Low-order methods, such as the finite-volume method and the finite-element method, usually provide linear convergence. Due to its mathematical simplicity and high accuracy with relatively few spatial grid points necessary, the CSM is considered an efficient technique in science and engineering computations, such as in computational fluid dynamics [17–19], electromagnetics [20], magnetohydrodynamics [21–23], reaction dynamics [24], and thermal radiation heat transfer [25–27]. Recently, Sun et al. [28] successfully developed the CSM to analyze convection and radiation heat transfer in a moving rod with temperature-dependent thermal conductivity.

**Nomenclature**

- \( A_{ij} \): entries of matrix A defined in Eq. (18)
- \( b_i \): coefficient of the integral term weight
- \( B_i \): entries of matrix B defined in Eq. (19)
- \( c_i \): coefficients of internal heat generation, W/m³
- \( ar{c}_i \): coefficient of the integral term weight
- \( G_i \): dimensionless coefficients of internal heat generation
- \( D_{ij}^{(1)} \): entries of the first-order derivative matrix
- \( D_{ij}^{(2)} \): entries of the second-order derivative matrix
- \( F_{ij} \): entries of matrix F defined in Eq. (21)
- \( G_i \): entries of matrix G defined in Eq. (22)
- \( h \): convective heat transfer coefficient, W/m² K
- \( h_b \): convective heat transfer coefficient corresponding to temperature difference \((T_b - T_\infty)\), W/m² K
- \( L \): length, m
- \( m_i \): adjustment parameters
- \( n \): exponent index
- \( N \): total number of collocation points
- \( N_{cc} \): convection-conduction parameter
- \( N_{rc} \): radiation-conduction parameter
- \( p_i \): Lagrange interpolation polynomials
- \( q \): volumetric heat generation rate, W/m³
- \( Q_f \): dimensionless fin heat transfer rate
- \( Q_{ideal} \): dimensionless ideal fin heat transfer rate
- \( R_{benchmark} \): benchmark results
- \( R_{CSM} \): CSM results
- \( s_i \): spectral collocation points
- \( T \): temperature, K
- \( T_b \): temperature at the fin base, K
- \( w_i \): entries of integral matrix defined in Eq. (24)
- \( x \): coordinate in x direction, m
- \( X \): dimensionless axial coordinate
- \( \alpha \): coefficient of thermal conductivity
- \( \beta \): coefficient of surface emissivity
- \( \delta \): half-thickness, m
- \( \delta_j \): parameter defined in Eq. (16)
- \( \varepsilon \): surface emissivity
- \( \varepsilon_{max} \): maximum relative error
- \( \eta \): fin efficiency
- \( \Theta \): dimensionless temperature
- \( \lambda \): thermal conductivity, W/m¹ K
- \( \sigma \): Stefan-Boltzmann constant, W/m² K

**Subscripts**

- \( i,j \): node indexes
- \( \infty \): value at the ambient temperature
In this treatise, the CSM is extended to solve nonlinear fin heat transfer with a wide range of temperature-dependent parameters including the internal heat generation, thermal conductivity, surface emissivity, and heat transfer coefficient. In the following, the physical model, the CSM formulation, and the corresponding solution procedure will be presented in Section 2. Validations of the method in comparison with analytical solutions in the literature will be carried out in Section 3. Effects of various significant parameters on fin heat transfer and efficiency will be discussed in Section 4. Finally, conclusions are summarized in Section 5.

2. Mathematical formulation

As shown in Figure 1, heat transfer in a rectangular straight fin with thickness 2δ, length L, and width W is considered. The fin surfaces lose heat through both convection and radiation. The thermal conductivity \( \lambda \), the surface emissivity \( \varepsilon \), the heat transfer coefficient \( h \), and the internal heat generation rate \( q \) in the fin are assumed temperature-dependent and can be expressed as

\[
\lambda(T) = \lambda_\infty \left[ 1 + a \left( \frac{T - T_\infty}{T_b - T_\infty} \right) \right]
\]

\[
\varepsilon(T) = \varepsilon_\infty \left[ 1 + \beta \left( \frac{T - T_\infty}{T_b - T_\infty} \right) \right]
\]

\[
h(T) = h_b \left( \frac{T - T_\infty}{T_b - T_\infty} \right)^n
\]

\[
q(T) = c_1 + c_2 \left( \frac{T - T_\infty}{T_b - T_\infty} \right) + c_3 \left( \frac{T - T_\infty}{T_b - T_\infty} \right)^2 + c_4 \left( \frac{T - T_\infty}{T_b - T_\infty} \right)^3
\]

where \( \lambda_\infty \) is the thermal conductivity at the ambient temperature \( T_\infty \), \( a \) is the coefficient of thermal conductivity; \( \varepsilon_\infty \) is the surface emissivity at the ambient temperature, \( \beta \) is the coefficient of surface emissivity; \( h_b \) is the heat transfer coefficient at the temperature difference between fin base \( T_b \) and ambient \( T_\infty \), \( n \) is the power index of the heat transfer coefficient, which depends on the mechanism of convection heat transfer; \( c_1 \) to \( c_4 \) are coefficients for the internal heat generation function. The radiation heat exchange between the fin and the fin base is neglected.

Based on a thin-fin assumption, the dimensionless energy equation of the fin per unit width can be expressed as [1]

\[
\frac{d}{dX} \left[ \left[ 1 + \alpha \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) \right] \frac{d\Theta}{dX} \right] - N_\infty \left( \Theta - \Theta_\infty \right)^{n+1} - N_c \left[ 1 + \beta \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) \right] \left( \Theta^4 - \Theta_\infty^4 \right) + \left[ C_1 + C_2 \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) + C_3 \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right)^2 + C_4 \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right)^3 \right] = 0
\]
where the dimensionless variables are defined as

\[ \Theta = \frac{T}{T_b} \quad \Theta_\infty = \frac{T_\infty}{T_b} \quad X = \frac{x}{L} \quad N_{cc} = \frac{h_b L^2 T_b^n}{\lambda_\infty \delta (T_b - T_\infty)^n} \quad N_{rc} = \frac{\sigma c_\infty L^2 T_b^3}{\lambda_\infty \delta} \quad C_i = \frac{c_i L^2}{\lambda_\infty T_b} \]  

(6)

The fin base is assumed a uniform temperature, while the fin tip is assumed adiabatic. Then, the dimensionless boundary conditions for Eq. (5) can be written as

\[ \Theta = 1 \quad X = 0 \]  

(7a)

\[ \frac{d\Theta}{dX} = 0 \quad X = 1 \]  

(7b)

Numerically, Eq. (5) can be rearranged as

\[
\left[ 1 + \alpha \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) \right] \frac{d^2 \Theta}{dX^2} + m_1 N_{cc} \Theta^{n+1} + m_2 N_{rc} \Theta^4 + m_3 C_3 \Theta^2 + m_4 C_4 \Theta^3 \\
= N_{cc} (\Theta - \Theta_\infty)^{n+1} + N_{rc} \left[ 1 + \beta \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) \right] (\Theta^4 - \Theta_\infty^4) - \frac{\alpha}{1 - \Theta_\infty} \left( \frac{d\Theta}{dX} \right)^2 \\
- \left[ C_1 + C_2 \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) + C_3 \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right)^2 + C_4 \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right)^3 \right] \\
+ m_1 N_{cc} \Theta^{n+1} + m_2 N_{rc} \Theta^4 + m_3 C_3 \Theta^2 + m_4 C_4 \Theta^3
\]  

(8)

where \( m_1 \) to \( m_4 \) are adjustment parameters. The values of adjustment parameters are subject to the convection-conduction parameter, radiation-conduction parameter, and internal heat generation coefficient.

Equation (8) and the corresponding boundary conditions [Eqs. (7a) and 7(b)] show that the distribution of dimensionless temperature in the fin depends on 10 parameters, namely, thermal conductivity coefficient \( \alpha \), exponent index of heat transfer coefficient \( n \), surface emissivity coefficient \( \beta \), convection-conduction parameter \( N_{cc} \), radiation-conduction parameter \( N_{rc} \), dimensionless ambient temperature \( \Theta_\infty \), and four dimensionless coefficients of internal heat generation \( C_i \).

Accordingly, the fin heat transfer rate is the sum of convective and radiative heat losses, i.e.,

\[ Q_f = \int_0^1 \left\{ N_{cc} (\Theta - \Theta_\infty)^{n+1} + N_{rc} \left[ 1 + \beta \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) \right] (\Theta^4 - \Theta_\infty^4) \right\} dX \]  

(9)

The ideal fin heat transfer rate is realized if the entire fin surface is maintained at the base temperature and can be obtained as

\[ Q_{\text{ideal}} = N_{cc} (1 - \Theta_\infty)^{n+1} + N_{rc} (1 + \beta)(1 - \Theta_\infty^4) \]  

(10)

The fin efficiency is defined as the ratio of the actual fin heat transfer rate to the ideal fin heat transfer rate,

\[ \eta = \frac{Q_f}{Q_{\text{ideal}}} = \frac{\int_0^1 \left\{ N_{cc} (\Theta - \Theta_\infty)^{n+1} + N_{rc} \left[ 1 + \beta \left( \frac{\Theta - \Theta_\infty}{1 - \Theta_\infty} \right) \right] (\Theta^4 - \Theta_\infty^4) \right\} dX}{N_{cc} (1 - \Theta_\infty)^{n+1} + N_{rc} (1 + \beta)(1 - \Theta_\infty^4)} \]  

(11)

To apply the CSM, the spatial distribution of temperature in the fin is discretized by spectral collocation points

\[ s_i = -\cos \left( \frac{\pi (i - 1)}{N - 1} \right) \quad i = 1, 2, \ldots, N \]  

(12)

where \( N \) is the number of collocation points.
The above spectral collocation points take values in the interval $[-1, 1]$. To meet the requirements of the spectral collocation points, a transformation equation should be used to map the real interval $(X: [0, 1])$ into the standard interval $(s: [-1, 1])$,

$$X = \frac{1}{2} (s + 1)$$

(13)

Using the CSM, the dimensionless temperature in the fin can be approximated by spectral collocation points and Lagrange interpolation polynomials,

$$\Theta(s) \approx \sum_{i=1}^{N} \Theta(s_i) p_i(s)$$

(14)

where $\{p_i(s)\}_{i=1}^{N}$ are Lagrange interpolation polynomials and can be obtained as

$$p_i(s) = \frac{(-1)^{i-1} \delta_i/(s-s_i)}{\sum_{j=1}^{N} (-1)^{j-1} \delta_j/(s-s_j)}$$

(15)

where

$$\delta_j = \begin{cases} 1/2 & j = 1, N \\ 1 & j = 2, 3, \ldots, N - 1 \end{cases}$$

(16)

Using this spectral discretization, the differential form of the energy equation is transformed to a matrix form of algebraic equations,

$$\sum_{i=1}^{N} A_{i,j} \Theta_j = B_i \quad i = 1, 2, \ldots, N$$

(17)

where the element expressions of matrices $A_{i,j}$ and $B_i$ are as follows:

$$A_{i,j} = \begin{cases} 4 \left[ 1 + \alpha \left( \frac{\Theta_i^* - \Theta_\infty}{1 - \Theta_\infty} \right) \right] D_{i,j}^{(2)} + m_1 N_{cc} (\Theta_i^*)^n \\ + m_2 N_{rc} (\Theta_i^*)^3 + m_3 C_3 (\Theta_i^*) + m_4 C_4 (\Theta_i^*)^2 \quad i = j \\ 4 \left[ 1 + \alpha \left( \frac{\Theta_i^* - \Theta_\infty}{1 - \Theta_\infty} \right) \right] D_{i,j}^{(2)} \quad i \neq j \end{cases}$$

(18)

$$B_i = N_{cc} (\Theta_i^* - \Theta_\infty)^{n+1}$$

$$+ N_{rc} \left[ 1 + \beta \left( \frac{\Theta_i^* - \Theta_\infty}{1 - \Theta_\infty} \right) \right] \left[ (\Theta_i^*)^4 - \Theta_i^4 \right] - \frac{4 \alpha}{1 - \Theta_\infty} \left( \sum_{j=1}^{N} D_{i,j}^{(1)} \Theta_j^* \right)$$

$$- C_1 + C_2 \left( \frac{\Theta_i^* - \Theta_\infty}{1 - \Theta_\infty} \right) + C_3 \left( \frac{\Theta_i^* - \Theta_\infty}{1 - \Theta_\infty} \right)^2 + C_4 \left( \frac{\Theta_i^* - \Theta_\infty}{1 - \Theta_\infty} \right)^3$$

$$+ m_1 N_{cc} (\Theta_i^*)^{n+1} + m_2 N_{rc} (\Theta_i^*)^4 + m_3 C_3 (\Theta_i^*)^2 + m_4 C_4 (\Theta_i^*)^3$$

(19)

The boundary conditions should be imported before solving the matrix equation. Imposing the Dirichlet boundary condition and the Neumann boundary condition, Eq. (17) becomes

$$\sum_{j=1}^{N-1} F_{ij} \Theta_{j+1} = G_i \quad i = 1, 2, \ldots, N - 1$$

(20)

where

$$F_{ij} = \begin{cases} A_{i+1,j+1} & i = 1, 2, \ldots, N - 2 \\ D_{N,j+1}^{(1)} & i = N - 1 \end{cases}$$

(21)
\[ G_i = \begin{cases} B_{i+1} - A_{i+1,1} \Theta_1 & i = 1, 2, \ldots, N - 2 \\ -D^{(1)}_{N,1} \Theta_1 & i = N - 1 \end{cases} \] (22)

Accordingly, the spectral discretized form of fin efficiency can be written as [14]

\[ \eta = \frac{\sum_{i=1}^{N} \left\{ N_{cc}(\Theta_i - \Theta_\infty)^{n+1} + N_{rc} \left[ 1 + \beta \left( \frac{\Theta_i - \Theta_\infty}{1 - \Theta_\infty} \right) \right] (\Theta_i^4 - \Theta_\infty^4) \right\} w_i}{N_{cc}(1 - \Theta_\infty)^{n+1} + N_{rc}(1 + \beta)(1 - \Theta_\infty^4)} \] (23)

where \( \{w_i\}_{i=1}^{N} \) are entries of the integral matrix and its detailed expression is [14]

\[ w_i = \frac{2}{N c_i} \sum_{j=\text{even}} b_j \cos \left( \frac{ij \pi}{N} \right) \] (24)

in which

\[ c_i = \begin{cases} 2 & j = 1, N \\ 1 & j = 2, 3, \ldots, N - 1 \end{cases} \quad \text{and} \quad b_j = \begin{cases} 0 & j = \text{odd} \\ 2/(1 - j^2) & j = \text{even} \end{cases} \]

The implementation of the CSM for solving the nonlinear fin heat transfer with multiple nonlinearities can be carried out according to the following routine.

**Step 1.** Choose the number of collocation points \( N \), and compute the spectral collocation points \( s_i \), the coordinate values \( X_i \), the first-order derivative matrix entries \( D^{(1)}_{i,j} \), the second-order derivative matrix entries \( D^{(2)}_{i,j} \), and the integral matrix entries \( w_i \).

**Step 2.** Make an initial guess for dimensionless temperature \( \Theta^* \).

**Step 3.** Compute the entries of spectral coefficient matrices \( A_{i,j} \) and \( B_j \) through Eqs. (18) and (19).

**Step 4.** Impose the Dirichlet and the Neumann boundary conditions; compute the entries of spectral coefficient matrices \( F_{i,j} \) and \( G_i \) by Eqs. (21) and (22).

**Step 5.** Directly solve the matrix form of algebraic equations (20) to get the new dimensionless temperature \( \Theta \).

**Step 6.** Terminate the iteration if the maximum relative difference between two consecutive iterations for dimensionless temperature is less than the convergence criterion (for example, \( 10^{-6} \)). Otherwise, go back to step 3.

**Step 7.** Calculate the fin efficiency by Eq. (23).

### 3. Validation of the CSM solution and Grid Independence

First, a special case with available analytical solution is considered to validate the above CSM. In this case, the thermal conductivity and heat transfer coefficient are assumed to be constant, and radiative heat transfer and internal heat generation are neglected. Thus, the energy equation [Eq. (5)] is reduced to

\[ \frac{d^2 \Theta}{dX^2} - N_{cc}(\Theta - \Theta_\infty) = 0 \] (25)

Then, the analytic solution of Eq. (25) can be derived as [1]

\[ \Theta(X) = \frac{1 - \Theta_\infty}{e^{2\sqrt{N_{cc}}} + 1} e^{\sqrt{N_{cc}}X} + \frac{(1 - \Theta_\infty)}{e^{2\sqrt{N_{cc}}} + 1} \frac{e^{-\sqrt{N_{cc}}X} \Theta_\infty}{e^{2\sqrt{N_{cc}}} + 1} + \Theta_\infty \] (26)

Computations of the CSM are executed on a computer with an Intel Core i5 2.40-GHz processor and 2.0 GB RAM memory. For the sake of quantitative comparison with analytical solution, the
maximum relative error is defined as

$$
\varepsilon_{\text{max}} = \max \left( \frac{|R_{\text{CSM}} - R_{\text{Benchmark}}|}{|R_{\text{Benchmark}}|} \right) \times 100\% 
$$

where $R_{\text{CSM}}$ is the CSM solution, and $R_{\text{Benchmark}}$ is the benchmark solution.

Figure 2 shows the dimensionless temperature distributions in the fin for three different values of convection-conduction parameter, namely, $N_{cc} = 0.2$, 5, and 15. The dimensionless ambient temperature is $\Theta_\infty = 0.2$. The total number of collocation points used for spatial discretization is $N = 15$. As shown in Figure 2, the CSM results have good agreement with the analytical results. The maximum relative error is less than 0.0007%.

In order to further validate the accuracy of the present method, the CSM is applied to a purely convective fin with temperature-dependent thermal conductivity and internal heat generation, and nonzero ambient temperature, to simulate the case studied by Ganji and Dogonchi [29]. A comparison of the CSM results with the DTM results [29] is shown in Figure 3. This comparison clearly indicates that the CSM results are very close to the DTM results, and the maximum relative error is 0.0002%.

Figure 2. Comparison of the CSM results with the exact analytical solution.

Figure 3. Comparison of the CSM results with the DTM results.
Figure 4 shows the maximum relative errors between the CSM results and the analytical solutions versus the total number of collocation points. Obviously, one can see that the CSM can provide very high accuracy with a small number of collocation points. The maximum relative errors decrease approximately following an exponential law with increase of collocation points until $N < 12$. As shown in Figure 4, the maximum relative error is less than $2.1064 \times 10^{-14}$ for $N_{cc} = 5$ when the total number of collocation points is $N = 15$. There is no further decrease in maximum relative error when increasing the total number of collocation points beyond. A similar trend is also observed for other convection-conduction parameters. Therefore, considering the computational economy and accuracy, $N = 15$ is used for spatial discretization in the following simulations.

4. Results and discussion

4.1. Effect of temperature-dependent thermal conductivity

Figure 5 shows the effect of temperature-dependent thermal conductivity on the dimensionless temperature distribution in the fin with $\alpha = -0.3$, 0.0, and 0.3, respectively. The remaining
parameters are $\beta = 0.3$, $n = 2$, $N_{cc} = 1.0$, $N_{rc} = 1.0$, and $\Theta_{\infty} = 0.3$. It is seen that the dimensionless temperature gradually increases with the increasing of thermal conductivity coefficient. This trend becomes more obvious at the fin tip. The reason is that for $\alpha = 0.3$, the thermal conductivity in the fin is proportional to the dimensionless temperature, and the conduction heat transfer is enhanced. For $\alpha = -0.3$, the phenomenon is just the contrary. Comparing the two different sets of internal heat generation coefficients, the dimensionless temperature gets higher when the internal heat generation increases in the fin. The variation of internal heat generation has more obvious impact on the dimensionless temperature at the fin tip.

Figures 6a and 6b illustrate effects of variable thermal conductivity on the fin efficiency versus convection-conduction parameter and radiation-conduction parameter, respectively. As shown in Figure 6a, the fin efficiency decreases as the convection-conduction parameter increases and as the coefficient of thermal conductivity decreases. To explain this phenomenon, it is noted that the convection-conduction parameter is the ratio of convection heat loss from the fin surface to conduction

![Figure 6](image_url). Effect of variable thermal conductivity on fin efficiency versus (a) convection-conduction parameter and (b) radiation-conduction parameter.
heat transfer in the fin. As $N_{cc}$ increases, it contributes to more convection heat loss from the fin surface, and results in a decrease in the fin efficiency. Increasing the coefficient of thermal conductivity can enhance conduction heat transfer and increase the dimensionless temperature, and consequently reduce the fin efficiency. Similarly, as seen in Figure 6b, increasing the radiation-conduction parameter and decreasing the coefficient of thermal conductivity can decrease the fin efficiency. Moreover, compared with the declining trend in Figure 6a, the trend in Figure 6b is more obvious. The reason is that radiation heat loss has more obvious impact on the fin efficiency, because radiation heat loss is proportional to the fourth power of the temperature difference between the ambient temperature and the surface temperature.

4.2. Effect of temperature-dependent surface emissivity

Figure 7 depicts the effect of emissivity coefficient on the dimensionless temperature distribution in the fin. The curves with $\beta > 0$ mean that the surface emissivity increases as dimensionless temperature
increases from $T_\infty$ to $T_b$. The converse is for curves with $\beta < 0$. As $\beta$ decreases from 0.3 to $-0.3$, the average surface emissivity decreases and so does the radiation heat loss from the surface of the fin. Therefore, the dimensionless temperature distribution in the fin increases. Also, because of decreased radiative heat loss, the magnitude of the dimensionless temperature gradient at the base, which is a measure of the fin heat transfer rate, must decreases as $\beta$ decreases.

### 4.3. Effect of temperature-dependent heat transfer coefficient

Figure 8 compares the temperature distributions in the fin with constant and temperature-dependent heat transfer coefficients to assess the effect of exponent index. The curves with solid circle symbols marked with $n = 0$ correspond to the temperature distribution assuming a constant heat transfer coefficient $h = h_b$; while the curves with hollow square symbols marked with $n = 2$ imply the temperature distribution with a nonlinear temperature-dependent heat transfer profile.

![Figure 8](image)

**Figure 8.** Temperature distribution with constant and temperature-dependent heat transfer coefficients.

![Figure 9](image)

**Figure 9.** Effects of the heat transfer coefficient and internal heat generation on the fin efficiency versus (a) convection-conduction parameter and (b) radiation-conduction parameter.
\[ h = h_b \left[ (T - T_\infty)/(T_b - T_\infty) \right]^2. \]

Compared with the constant heat transfer coefficient \( n = 0 \), the temperature-dependent heat transfer coefficient \( n = 2 \) results in higher dimensionless temperature and can produce more uniform temperature distribution along the fin. Figure 8 also shows that the dimensionless temperature distribution is strongly dependent on dimensionless internal heat generation parameters. It is observed that the dimensionless temperature gets higher when internal heat generation goes up.

Figure 9a shows the effects of convection-conduction parameter and parameter of heat transfer coefficient on the fin efficiency. It is seen that for constant heat transfer coefficient \( n = 0 \), the fin efficiency decreases as the convection-conduction parameter increases; and a similar trend is found for the case of temperature-dependent heat transfer coefficient \( n = 2 \). According to Eq. (6), the increase in convection-conduction parameter would be due to the increase in \( h_b \), and consequent the fin efficiency reduces. Compared with the fin heat transfer without internal heat generation, the fin heat transfer with internal heat generation exhibits higher fin efficiency. As explained in regard
to Figure 8, the internal heat generation can increase the dimensionless temperature distribution, and enhance the fin efficiency.

Similarly, Figure 9b illustrates effects of radiation-conduction parameter and parameter of heat transfer coefficient on the fin efficiency. As shown in Figure 9b, for both constant heat transfer coefficient and variable heat transfer coefficient, the fin efficiency decreases with the increasing of the radiation-conduction parameter. Compared with the case of constant heat transfer coefficient \((n = 0)\), the fin heat transfer with temperature-dependent coefficient \((n = 2)\) is more efficient.

### 4.4. Effects of convection-conduction and radiation-conduction parameters

In Figures 10 and 11, the effects of the convection-conduction parameter \(N_{cc}\) and radiation-conduction parameter \(N_{rc}\) on the dimensionless temperature distribution are given for \(\alpha = 0.3\), \(\beta = 0.3\), \(n = 2\), and \(\Theta_\infty = 0.3\). It is seen that for a constant \(N_{rc}\), variable \(N_{cc}\) imposes a significant impact on the dimensionless temperature distribution; and for a given \(N_{cc}\), \(N_{rc}\) also exerts a major effect on the dimensionless temperature distribution. \(N_{cc}\) is defined to be the ratio of convection heat loss from the fin surface to conduction heat transfer in the fin. Similarly, \(N_{rc}\) is defined as the ratio of radiation heat loss from the fin surface to conduction heat transfer in the fin. As the \(N_{cc}\) increases, it leads to more heat loss from the fin surface. Hence, the dimensionless temperature distribution becomes steeper from left to right. Figures 10 and 11 also demonstrate that internal heat generation can increase the dimensionless temperature in the fin. Furthermore, comparing Figures 10 and 11, one can see that \(N_{rc}\) imposes more obvious effect on the dimensionless temperature distribution than \(N_{cc}\) does.

### 4.5. Effect of dimensionless ambient temperature

Figure 12 illustrates the effect of dimensionless ambient temperature on the dimensionless temperature distribution in the fin. As the dimensionless ambient temperature increases, radiation and convection heat loss decreases, and consequently the dimensionless temperature increases.

Figures 13a and 13b display the effects of convection-conduction parameter and radiation-conduction parameter together with dimensionless ambient temperature on the fin efficiency, respectively. As shown in Figures 13a and 13b, the fin efficiency slightly increases as dimensionless ambient temperature increases.

![Figure 12. Dimensionless temperature distributions for three different ambient temperatures and two different sets of coefficients of internal heat generation.](image-url)
5. Conclusions

A nondimensional mathematical model describing nonlinear heat transfer in a straight fin with temperature-dependent properties has been introduced and solved by the CSM. The CSM results have been compared with exact solutions and available numerical results in the literature, and excellent agreement was found. The maximum relative error between the CSM results and the analytical solutions approximately follows an exponential decay as the total number of collocation points increases. A small number of collocation points is needed to obtain accurate results; and thus, the method is very efficient in computation.

Numerical results show that a fin is more effective when the fin material has a higher thermal conductivity coefficient. The fin temperature and the fin efficiency increase with the increasing of thermal conductivity coefficient. Similarly, the fin temperature increases as the coefficient of surface emissivity increases. As the convection-conduction parameter or the radiation-conduction parameter increases, the distribution of fin temperature becomes steeper from the fin base to the fin tip; and the
radiation-conduction parameter is more significant. The fin temperature gradually increases as the value of ambient temperature increases. The collocation of fin temperature and fin efficiency graphs would be useful in the design of a variety of engineering systems where fins are adopted to enhance heat transfer.

Conflict-of-interest statement

Authors state that the manuscript does not have any conflict of interest including any financial, personal, or other relationships with other people or organizations within three years of beginning the submitted work that could inappropriately influence, or be perceived to influence, the present work.

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