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Improved Treatment of Anisotropic Scattering in Radiation Transfer Analysis Using the Finite Volume Method

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Discretization of the integral anisotropic-scattering term in the equation of radiative transfer will result in two kinds of numerical errors: alterations in scattered energy and asymmetry factor. Though quadrature flexibility with large angular directions and further solid-angle splitting in the finite volume method (FVM) allow for reduction/minimization of these errors, computational efficiency is adversely impacted. A phase-function normalization technique to get rid of these errors is simpler and is applied to the three-dimensional (3-D) FVM for the first time to improve anisotropic radiation transfer computation accuracy and efficiency. FVM results are compared to Monte Carlo and discrete-ordinates method predictions of radiative heat transfer in a cubic enclosure housing a highly anisotropic participating medium. It is found that the FVM results generated using the normalization technique conform accurately to the results of the other two methods with little impact on computational efficiency.

INTRODUCTION

In engineering problems where radiation is the dominant mode of heat transfer, such as high-temperature combustion and material processing [1–5], fire and solar radiation [6–8], and laser applications [9–12], accurate solutions of the equation of radiative transfer (ERT) are required for full radiation characterization. The integro-differential nature of the ERT makes analytical solution difficult, and thus numerical methods, such as the finite volume method (FVM) and discrete-ordinates method (DOM), are preferred.

The FVM was introduced as a method of predicting radiation heat transfer in the early 1990s [13–16]. Raithby [17] presented an excellent discussion of the FVM in two-dimensional (2-D) and three-dimensional (3-D) enclosures with unstructured grids. Chai et al. [18] analyzed ultrafast radiation heat transfer in a 3-D rectangular medium using the FVM, expanding on an earlier similar analysis using the DOM [19]. Kim and Huh [20] introduced a new angular discretization scheme for the FVM.

In practical applications, radiation scattering is anisotropic. In all discretization-based numerical methods, including FVM and DOM, the continuous angular variation of radiation scattering is approximated using a finite number of discrete radiation directions. A well-known issue for anisotropic-scattering media is that scattered energy becomes nonconserved after directional discretization [21]. This issue can be corrected using phase-function normalization in DOM [21–26]. For the FVM, scattered energy may be conserved using a solid-angle splitting technique [14]. A less known issue is the alteration in phase-function asymmetry factor [22–27] for both FVM and DOM, resulting in an error termed “angular false scattering.”

Recently, Hunter and Guo [22] developed a phase-function normalization technique to simultaneously conserve scattered energy and phase-function asymmetry factor after directional discretization. The impact of this normalization on angular false-scattering errors was analyzed for 2-D axisymmetric [22, 23] and 3-D cubic enclosures [24, 25] using the DOM, where DOM predictions were vastly improved in comparison to benchmark Monte Carlo (MC) results. Hunter and Guo [27] found that...
angular false scattering still persisted after FVM discretization even with substantial refinement in solid-angle splitting in 2-D axisymmetric cylindrical enclosures. As most practical applications cannot be approximated as two-dimensional, it is necessary to investigate angular false-scattering errors for 3-D FVM.

In this study, radiation transfer in a 3-D cubic enclosure containing an anisotropic-scattering medium is predicted using the FVM. The necessity of using the authors’ phase-function normalization technique to ensure minimization of angular false scattering and improve treatment of anisotropic scattering in radiation transfer analysis is presented. Heat fluxes generated using the FVM both with and without phase-function normalization are compared to MC [26] and DOM results [25] to gauge the accuracy of the FVM predictions. The impact of solid-angle splitting on both scattered energy and asymmetry factor conservation are compared to MC [26] and DOM results [25] to gauge the accuracy of the FVM predictions. A discussion on the computational advantages of using phase-function normalization is presented.

**THE FINITE VOLUME METHOD**

The steady-state ERT of radiation intensity in a gray, absorbing-emitting and anisotropically scattering medium can be expressed, using general vector notation, as follows:

\[ \mathbf{s} \cdot \nabla I(\mathbf{r}, \mathbf{s}) = -(\sigma_a + \sigma_s)I(\mathbf{r}, \mathbf{s}) + \sigma_a I_p(\mathbf{r}) + \frac{\sigma_s}{4\pi} \int \int I(\mathbf{r}, \mathbf{s}') \Phi(\mathbf{s}', \mathbf{s}) d\Omega' \]  

In the preceding equation, the term on the left-hand side accounts for spatial gradients of radiative intensity, while the three right-hand-side terms represent intensity attenuation due to both absorption and radiative out-scattering, intensity augmentation due to medium emission, and intensity augmentation due to in-scattering of radiative energy, respectively.

Using a control-volume approach, Eq. (1) is integrated over control volume \( \Delta V \) and discrete solid angle \( \Delta \Omega' \) [16], defined by azimuthal angle \( \phi \) and polar angle \( \theta \), where the discrete radiation direction \( \mathbf{s}' \) denotes the centroid of \( \Delta \Omega' \) [15, 16]. After performing the integration and evaluating the integrals over control volume and solid angle, the discretized form of Eq. (1) can be expressed as follows:

\[ \sum_i I^i D^i = -(\sigma_a + \sigma_s) \Delta V \Delta \Omega' + S^i \Delta V \Delta \Omega', \quad i = 1, 2, \ldots, M \]  

where \( M \) is the total number of discrete radiation directions \( \mathbf{s}' \). In this summation, \( I^i \) represents radiative intensity in discrete direction \( \mathbf{s}' \) at control-volume face \( i \), \( A_i \) is the facial surface area of control-volume face \( i \), and \( D^i \) is the directional weight of discrete direction \( \mathbf{s}' \) at control-volume face \( i \), evaluated using the following expression:

\[ D^i = \int \int (\mathbf{s} \cdot \mathbf{n}) d\Omega \]  

where \( \mathbf{n} \) is the unit vector normal to control-volume face \( i \).

The source term \( S^i \) can be expressed as

\[ S^i = \sigma_a I_p + \frac{\sigma_s}{4\pi} \sum_{l=1}^{M} \Phi^{il} \Delta \Omega^l \]  

where the radiative in-scattering integral in Eq. (1) has been approximated using a discrete quadrature summation. In this summation, \( \Phi^{il} \) is the average scattering phase function between two discrete solid angles \( \Delta \Omega^l \) and \( \Delta \Omega^i \), which can be calculated as such:

\[ \Phi^{il} = \frac{1}{\Delta \Omega^l \Delta \Omega^i} \int \int \Phi(\mathbf{s}', \mathbf{s}) d\Omega d\Omega' \]  

The necessity of using an averaged scattering phase function will be described later.

The choice of directions for the FVM is generally arbitrary. Commonly, the total solid angle of \( 4\pi \) is approximated using \( M = (N_\phi \times N_\theta) \) discrete directions, where \( N_\phi \) and \( N_\theta \) are the number of divisions in the azimuthal and polar angle, respectively. However, Kim and Huh [20] discovered that for 3-D problems, angular discretization in this manner resulted in highly nonuniform solid angles. Thus, they introduced a new angular discretization method for the FVM called the \( FT_N \)-FVM. Using this method, the total polar angle \( \theta \) is divided into an even number \( N \) of uniformly spaced directional levels. The number of azimuthal divisions corresponding to the \( N \) polar levels follows the arithmetic sequence 4, 8, 12, \ldots, 2\( N - 4 \), 2\( N \), 2\( N - 4 \), \ldots, 12, 8, 4. The total number of directions becomes \( M = N (N + 2) \). Kim and Huh [20] showed that this procedure produces more uniform solid angles, leading to improvement in radiation transfer results over the commonly used \( (N_\phi \times N_\theta) \) angular discretization. To this end, the \( FT_N \)-FVM discretization procedure is implemented for all forthcoming results in this analysis.

Additional details on the discretization of the ERT and solution procedure using the FVM are not presented here, for brevity, but are available in a textbook [1] and in the authors’ recent publications [23, 28].

**PHASE FUNCTION NORMALIZATION**

It is well established that discretization of the angular variation must conserve scattered energy for all discrete directions \( \mathbf{s}' \):

\[ \frac{1}{4\pi} \sum_{l=1}^{M} \Phi^{il} \Delta \Omega^l = 1 \]  

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If scattering is isotropic, the preceding condition is explicitly satisfied. However, radiation scattering is always anisotropic in natural materials and the preceding conservation is usually broken when anisotropy is considered [14]. Inaccurate conservation of scattered energy has been shown to produce inaccurate radiation transfer predictions [26].

To correct this issue, Chui et al. [14] introduced a solid-angle splitting technique, in which each solid angle $\Delta \Omega^j$ is subdivided into multiple subangles $\Delta \Omega^i$. The total scattered energy between any two arbitrary solid angles is then calculated by averaging the energy scattered between their corresponding sub-angles. Using this technique, the average scattering phase function of Eq. (5) can be approximated as follows [13]:

$$\Phi^{fi} \approx \frac{1}{\Delta \Omega^j} \sum_{i=1}^{M_s} \sum_{i'=1}^{M_s'} \Phi^{fi,i'} \Delta \Omega^j \Delta \Omega^i$$

(7)

In the preceding equation, $\Phi^{fi,i'}$ is the discrete scattering phase function between subangles $\Delta \Omega^j$ and $\Delta \Omega^i$, and $M_s$ and $M_s'$ are the number of total subangles in solid angles $\Delta \Omega^j$ and $\Delta \Omega^i$, respectively. After averaging, the scattered energy conservation of Eq. (6) becomes

$$\frac{1}{4\pi} \sum_{i=1}^{M_s} \Phi^{fi,i} \Delta \Omega^i = 1$$

(8)

Assuming every solid angle is divided into a sufficient number of subangles, this conservation condition will be accurately satisfied for all directions $\hat{s}^j$, regardless of phase-function type.

In order to ease numerical computation, approximations to the highly oscillatory Mie phase function $\Phi$ are commonly implemented, as the physical nature of $\Phi$ makes it difficult to efficiently adopt it. For highly anisotropic scattering, the Heney–Greenstein (HG) phase function is widely accepted as a suitable approximation, due to its accurate representation of the strong forward-scattering peak [1]. The HG phase function approximation can be expressed as

$$\Phi_{HG}(\theta) = \sum_{n=0}^{\infty} (2n+1) g^n P_n(\cos \theta)$$

$$= \frac{1 - g^2}{[1 + g^2 - 2g \cos \Theta]^{1.5}}$$

(9)

where $\Theta$ is the scattering angle between incoming and outgoing radiation directions, $P_n$ is the $n$th-order Legendre polynomial, and the phase-function asymmetry factor $g$ is the average cosine of scattering angle.

Figure 1 illustrates the dependence of scattered energy conservation on the number of solid-angle splitting after FVM discretization for HG phase function approximation with $g = 0.9300$. The continuous angular variation is discretized using $M = 24, 48, 80, 168$, and 288 discrete directions. Each solid angle is subdivided into $(N_{\phi} \times N_{\theta})$ subangles, with $N_{\phi} = N_{\theta}$, ranging from $(2 \times 2)$ to $(24 \times 24)$ divisions. For a given number of sub-angles, increases in the number of subangles reduce the discrepancy in discretized scattered energy conservation. In order to conserve scattered energy accurately within 0.001%, $(N_{\phi} \times N_{\theta}) = (24 \times 24), (20 \times 20), (20 \times 20), (12 \times 12)$, and $(12 \times 12)$ subangles are required for $M = 24, 48, 80, 168$, and 288, respectively. Sufficient solid-angle splitting may accurately conserve scattered energy.

In addition to scattered energy, the overall phase-function asymmetry factor $g$ should also be conserved after directional discretization [22]. Thus, the following relation should hold for all discrete directions $\hat{s}^j$:

$$\frac{1}{4\pi} \sum_{i=1}^{M_s} \Phi^{fi,i} \cos \theta^{i'} \Delta \Omega^j = g$$

(10)

Figure 2 illustrates the deviation from asymmetry factor conservation for the same conditions as Figure 1. The behaviors are quite different from Figure 1. When solid-angle splitting level is initially increased, the discretized asymmetry factor value will converge toward its prescribed value ($g = 0.9300$); however, the situation will worsen with further increase of splitting level after passing the convergence. At a very high splitting level of $(N_{\phi} \times N_{\theta}) = (24 \times 24)$, the discretized asymmetry factor for $M = 24, 48, 80, 168$, and 288 attains a value of 0.8855, 0.9024, 0.9113, 0.9198, and 0.9237, respectively. This discrepancy between prescribed and discretized asymmetry factor in the FVM had gone largely unnoticed, as the commonly implemented solid-angle splitting technique was assumed to accurately conserve the asymmetry factor [26] as well as scattered energy. Small errors in discretized asymmetry factor can produce significant errors in radiation transfer predictions. Errors of
Figure 2 Deviation from asymmetry factor conservation versus solid-angle splitting number for HG phase function with \( g = 0.9300 \) with various angular quadratures.

this type are really false scattering due to angular discretization, or termed “angular false scattering” [24, 25].

To accurately conserve both scattered energy and asymmetry factor simultaneously, the average scattering phase function is normalized in the following manner [22]:

\[
\tilde{\Phi}^{ij} = \left(1 + A^{ij}\right) \Phi^{ij}
\]

where the normalization parameter \( A^{ij} \) corresponds to scattering between two discrete directions \( \hat{s}^i \) and \( \hat{s}^j \). Normalization parameters \( A^{ij} \) are determined such that \( \tilde{\Phi}^{ij} \) satisfies Eqs. (8) and (10), as well as directional symmetry (\( \tilde{\Phi}^{ij} = \tilde{\Phi}^{ji} \)). Values of normalization matrix \( A^{ij} \) that will accurately conserve scattered energy as well as asymmetry factor after FVM discretization can be determined using pseudo-inversion.

RESULTS AND DISCUSSION

The benchmark test problem involves radiation heat transfer in a cubic participating enclosure of edge length \( L \) shown in Figure 3. The spatial coordinates are nondimensionalized as follows: \( x^* = x/L, \ y^* = y/L, \) and \( z^* = z/L \). The optical thickness and scattering albedo of the medium are \( \tau = (\sigma_a + \sigma_s) L \) and \( \omega = \sigma_s/(\sigma_a + \sigma_s) \). Unless otherwise specified, the medium is taken to be cold \( (I_o = 0) \) and purely scattering \( (\omega = 1.0) \) with an optical thickness \( \tau = 10.0 \). All boundary walls are black, with the wall at \( z^* = 0 \) taken as a diffuse emitter with unity emissive power, and all remaining walls are taken to be cold. In order to relate the intensities at control-volume nodes to that of the control-volume faces in the FVM solution scheme, the positive spatial differencing scheme is used. For all simulations, a staggered spatial control-volume grid of \( (N_x \times N_y \times N_z) = (27 \times 27 \times 27) \) with uniform grid steps of \( \Delta x^* = \Delta y^* = \Delta z^* = 0.04 \) is implemented, in order to minimize spatial discretization error.

The computing workstation used is a Dell Optiplex 780, with an Intel 2 Dual Core 3.16 GHz processor and 4.0 GB of RAM. The FVM procedure was implemented using the FORTRAN computing language, and the values of the normalization parameters were determined by using MATLAB, and imported into FORTRAN.

Figure 4 examines the impact of phase-function normalization on heat flux \( Q(x^*, y^* = 0.5, z^* = 1.0) \) for three representative HG asymmetry factors: \( g = 0.2000, 0.8000, \) and \( 0.9300 \) generated using FVM with \( M = 168 \) for \( g = 0.9300 \) and comparison with MC solutions [26].
Figure 5  Impact of discrete direction number on $Q(x^*,y^*=0.5,z^*=1)$ and comparison with MC results [26] for $g = 0.9300$.

0.9300. Profiles generated via the FVM using either Hunter and Guo’s normalization technique or no normalization with sufficient solid-angle splitting are compared to reference MC results [26]. The FVM profiles were generated using $M = 168$ discrete directions, and thus solid-angle splitting of $(Ns \times Ns) = (16 \times 16)$ is implemented so that scattered energy is conserved even without phase-function normalization. For weakly forward scattering ($g = 0.2000$), FVM profiles both with and without phase-function normalization produce nearly identical results. As asymmetry factor is increased to $g = 0.8000$, nonnormalization leads to a 5% change in scaled scattering effect $(1 - g)$. For the highly anisotropic case, the discretized $g$ is altered from 0.9300 to 0.9198 (14.6% scaled scattering effect change). When compared to MC predictions, heat fluxes generated using normalization conform more accurately than when normalization is ignored. Without normalization, the FVM profile even with a large number of solid-angle splitting underpredicts the MC by 10%.

Figure 5 investigates the impact of angular discretization on both normalized and non-normalized FVM heat flux in a highly anisotropic medium with $g = 0.9300$. Reference MC values are also plotted, for comparison. Solid angle splitting of $(24 \times 24)$ for $M = 24$, $(20 \times 20)$ for $M = 48$ and 80, and $(16 \times 16)$ for $M = 168$ is applied to both normalized and nonnormalized profiles, in order to accurately conserve scattered energy even without normalization. It is seen that normalization is much more efficient than increasing the angular directions in reducing the angular false-scattering errors. When normalization is applied, the differences between FVM and MC are dramatically reduced. One exception exists for the lowest directional order $M = 24$, where a difference of near 30% is still witnessed near the wall center.

The underlying cause of this large discrepancy may be found by examining the distribution of discretized phase function versus the cosine of scattering angle in comparison with the theoretical HG phase function values calculated using Eq. (9).

As shown in Figure 6a for $M = 24$, although asymmetry factor is effectively conserved after phase-function normalization, the discretized values of scattering phase function still exhibit some differences from the theoretical phase function values. The minimal amount of discrete directions $M = 24$ in the lowest quadrature isn’t able to accurately represent the true nature of the theoretical phase function shape. In contrast, when an intermediate number of discrete directions is implemented, for
example, $M = 168$ as shown in Figure 6b, both asymmetry factor and phase-function shape are preserved after normalization.

A comparison of heat flux profile $Q(x^*, y^* = 0.5, z^* = 1.0)$ generated using the FVM both with and without phase-function normalization to both MC [26] and DOM $S_{12}$ [25] results is presented in Table 1. For DOM, there is no way to implement an analogous solid-angle splitting technique, so phase-function normalization is always required for anisotropic scattering radiation transfer. Both DOM and FVM adopt the same spatial grid, that is, $(N_x \times N_y \times N_z) = (27 \times 27 \times 27)$, and have the same directional number $M = 168$. Solid-angle splitting of $(24 \times 24)$ is used in the FVM. Results should be similar among the three solution methods if they are accurate. Heat fluxes generated using the DOM $S_{12}$ with Hunter and Gu’s normalization technique, which accurately preserves the prescribed $g$ value, result in differences of $<7\%$ at all locations in comparison to MC. Sufficient solid-angle splitting in FVM to conserve scattered energy produces a discretized $g = 0.9198$ without normalization, resulting in underpredictions up to $12\%$ as compared with MC. Application of normalization in the FVM accurately preserves the prescribed $g$ value and reduces the error to $<5\%$ at maximum. The average difference between DOM and FVM with normalization is only $3\%$. The accurate conformity of normalized FVM to both MC and normalized DOM results gives confidence that anisotropic scattering properties of the medium are being accurately accounted for through proper phase-function normalization. The impact of solid-angle splitting on radiation heat flux results generated using the FVM both with and without normalization is examined in Figures 7a and 7b for $M = 80$ and 288, respectively, in which percentage differences between FVM heat fluxes and reference MC results are plotted for various levels of solid-angle splitting. As discussed in the results shown in Figure 1, solid-angle splitting of $(N_{\phi} \times N_{\omega}) = (20 \times 20)$ and $(12 \times 12)$ are required to accurately conserve scattered energy within 0.001$\%$ for $M = 80$ and 288, respectively. As seen in Figure 7, there exist large heat flux percentage differences when normalization is ignored for these two cases. Increasing the splitting level to $(24 \times 24)$, the discrepancies between MC and FVM results without normalization are still large. However, the differences between normalized FVM and MC are very small, even with use of the lowest splitting level of $(2 \times 2)$, FVM profiles with normalization generated with both lowest $(2 \times 2)$ and high splitting $(24 \times 24)$ densities are nearly identical. This indicates that further solid-angle splitting past $(2 \times 2)$ is not required to obtain more accurate radiation transfer solutions when implementing normalization.

The ability to produce accurate FVM solutions with minimal solid-angle splitting and low-order directional quadrature has a distinct advantage when it comes to computational convergence time in addition to memory requirement. FVM convergence times for varying number of discrete directions and various solid angle splitting levels are presented in Table 2 for the problem analyzed in Figure 5. Computational times for

Table 1: Comparison of $Q(x^*, y^* = 0.5, z^* = 1.0)$ values generated using normalized DOM [25] and FVM to MC values [26] for a 3-D cube with $\tau = 10, \omega = 1$, and $g = 0.93$

<table>
<thead>
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<th>x/L</th>
<th>DOM Normalization</th>
<th>No Normalization</th>
<th>FVM Normalization</th>
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<td>0.02</td>
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<td>0.0930</td>
<td>0.1005</td>
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<tr>
<td>0.50</td>
<td>0.1565</td>
<td>0.1558</td>
<td>0.1633</td>
</tr>
</tbody>
</table>

Table 2: Computational convergence times, in seconds, for FVM with and without normalization at various solid-angle splitting levels and varying number of discrete directions

<table>
<thead>
<tr>
<th>M</th>
<th>(2 × 2)</th>
<th>(4 × 4)</th>
<th>(6 × 6)</th>
<th>(8 × 8)</th>
<th>(12 × 12)</th>
<th>(16 × 16)</th>
<th>(20 × 20)</th>
<th>(24 × 24)</th>
<th>(2 × 2)</th>
</tr>
</thead>
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<tr>
<td>24</td>
<td>Diverge</td>
<td>Diverge</td>
<td>21.37</td>
<td>14.70</td>
<td>15.38</td>
<td>20.76</td>
<td>33.03</td>
<td>50.86</td>
<td>14.51</td>
</tr>
<tr>
<td>48</td>
<td>Diverge</td>
<td>75.21</td>
<td>50.00</td>
<td>47.21</td>
<td>54.40</td>
<td>77.532</td>
<td>124.0</td>
<td>207.0</td>
<td>41.84</td>
</tr>
<tr>
<td>80</td>
<td>Diverge</td>
<td>116.5</td>
<td>110.6</td>
<td>113.0</td>
<td>141.8</td>
<td>203.9</td>
<td>355.9</td>
<td>556.6</td>
<td>109.8</td>
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<tr>
<td>168</td>
<td>Diverge</td>
<td>531.0</td>
<td>491.5</td>
<td>514.1</td>
<td>606.7</td>
<td>834.2</td>
<td>1415</td>
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<td>2438</td>
<td>4133</td>
<td>7011</td>
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</table>
Figure 7 Percent difference in $Q(x^*, y^* = 0.5, z^* = 1.0)$ from MC solutions between MC solutions [26] and FVM solutions both with and without phase-function normalization using various solid angle splitting densities for a) $M = 80$ and b) $M = 288$.

Figure 8 Comparison of $Q(x^*, y^* = 0.5, z^* = 1.0)$ between MC solutions [26] and FVM solutions using high-order quadrature.

Figure 9 plots FVM heat fluxes, generated with and without normalization, in an optically thin $(1 - g \tau = 0.07)$, purely scattering medium with $g = 0.9300$ in order to gauge the impact of optical thickness. MC results taken from Boulet et al. [26] are also presented for comparison. Sufficient solid-angle splitting results in a 406% increase in computational time. Use of normalization allows for convergence and accuracy with just $(2 \times 2)$ splitting and relatively low-order quadrature, reducing computational times substantially.

Based on the trend seen in Figure 2, further increase in discrete direction number should reduce the discrepancy between discretized and prescribed asymmetry factor for high solid-angle splitting without phase-function normalization. It therefore is possible, with a sufficient number of discrete directions, that scattered energy and asymmetry factor could be accurately conserved without additional phase-function normalization. This is investigated in Figure 8, wherein FVM heat fluxes generated using up to 2024 discrete directions without the use of phase-function normalization are plotted versus wall location $x^*$. Solid-angle splitting of $(16 \times 16)$ for $M = 168$ and 288, $(8 \times 8)$ for $M = 624$ and 1088, and $(4 \times 4)$ for $M = 2024$ is applied to accurately conserve scattered energy. As a comparison, both reference MC solutions and FVM solutions with normalization at $M = 168$ are also plotted. Increasing discrete direction number to $M = 624, 1088,$ and $2024$ results in discretized asymmetry factors of $g = 0.9270, 0.9282,$ and $0.9290,$ respectively, moving closer to the prescribed value $g = 0.93$. When corresponding FVM heat fluxes are compared to MC, it is seen that an increase in directions does reduce differences of results between MC and FVM. However, the three profiles with $M = 624, 1088,$ and $2024$ took 5650, 16,520, and 65,400 seconds to converge, respectively. In addition, the FVM profile using normalization with $M = 168$ and $(2 \times 2)$ splitting is more accurate than these extreme direction cases, and took only 497 seconds.
normalization is ignored. While sufficient increases in discrete
direction number can reduce/minimize angular false-scattering
errors in FVM, use of phase-function normalization is a more
effective method to improve treatment of anisotropic scattering
in radiation transfer computation, as it improves computational
efficiency and accuracy.

CONCLUSIONS

The commonly implemented technique of solid-angle split-
ing in the FVM is able to conserve scattered energy, provided
that a sufficient splitting density is used. Nevertheless, it may
not be able to preserve the phase-function asymmetry factor,
leading to angular false scattering. When scattering is highly
anisotropic, small deviations in asymmetry factor result in sig-
nificant discrepancies in FVM radiation predictions when com-
pared to reference MC results. Application of proper phase-
function normalization eliminates errors in discretized asym-
metry factor, as well as scattered energy, and produces FVM
results that are more accurate to MC predictions than when

Figure 9 Impact of discrete direction number on \(Q(x^*, y^* = 0.5, z^* = 1)\) and comparison with MC results [26] for \(g = 0.9300\) in an optically-thin medium.

is implemented to conserve scattered energy for nonnormalized
profiles, while for normalized FVM minimum \((2 \times 2)\) splitting is
used. For the four direction numbers presented \((M = 48, 80, 168,
and 288)\), the difference between nonnormalized and normalized
FVM heat fluxes is minimal. For an optically thin medium,
radiative energy is able to penetrate a much larger distance into
the medium before scattering. Therefore, fewer scattering events
occur as compared to the relatively thick medium described in
the rest of the analysis, and the distortions in asymmetry factor
when normalization is not applied do not have as strong an
impact. In addition, the FVM heat flux profiles with low-order
quadrature \((M = 48 \text{ and } 80)\) exhibit different behavior and shape
than the reference MC profiles. Physically impossible bumps in
the heat flux profiles appear due to ray effect [29], an error based
on angular discretization that has been shown to be prominent for
optically thin media. Thus, while phase-function normalization
minimizes angular false scattering errors, ray effect could be still
prominent. As direction number is increased, ray effect becomes
mitigated in both normalized and nonnormalized FVM results.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>(A_i)</td>
<td>facial surface area of CV face (i)</td>
</tr>
<tr>
<td>(A_l^f)</td>
<td>normalization coefficients</td>
</tr>
<tr>
<td>(D_i^l)</td>
<td>directional weight at face (i) in direction (l)</td>
</tr>
<tr>
<td>DOM</td>
<td>discrete ordinates method</td>
</tr>
<tr>
<td>ERT</td>
<td>equation of radiation transfer</td>
</tr>
<tr>
<td>FVM</td>
<td>finite volume method</td>
</tr>
<tr>
<td>(g)</td>
<td>asymmetry factor</td>
</tr>
<tr>
<td>HG</td>
<td>Henyey–Greenstein</td>
</tr>
<tr>
<td>(I)</td>
<td>radiative intensity ((W/m^2sr))</td>
</tr>
<tr>
<td>(L)</td>
<td>edge length of cubic enclosure</td>
</tr>
<tr>
<td>(M)</td>
<td>total number of discrete directions</td>
</tr>
<tr>
<td>MC</td>
<td>Monte Carlo</td>
</tr>
<tr>
<td>(N_{\theta}, N_{\phi})</td>
<td>number of divisions in polar and azimuthal direction</td>
</tr>
</tbody>
</table>
| \(N_{s_{\theta}}, N_{s_{\phi}}\) | solid-angle subdivisions in polar and azimuthal di-
| \(\hat{n}\) | surface outward normal vector |
| \(P_n\) | the \(n\)-th order Legendre polynomial |
| \(Q\) | nondimensional heat flux |
| \(r\) | position vector |
| \(r\) | radial location |
| \(S\) | radiation source term |
| \(\hat{s}\) | unit direction vector |
| \(x, y, z\) | Cartesian coordinates |
| \(x^*, y^*, z^*\) | dimensionless Cartesian coordinates |

Greek Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta A, \Delta V)</td>
<td>control volume surface area and volume ((m^2, m^3))</td>
</tr>
<tr>
<td>(\Delta \Omega)</td>
<td>discrete solid angle ((sr))</td>
</tr>
<tr>
<td>(\sigma_{\theta})</td>
<td>absorption coefficient (\left(m^{-1}\right))</td>
</tr>
<tr>
<td>(\sigma_{\phi})</td>
<td>scattering coefficient (\left(m^{-1}\right))</td>
</tr>
<tr>
<td>(\Phi)</td>
<td>scattering phase function</td>
</tr>
<tr>
<td>(\tilde{\Phi})</td>
<td>normalized scattering phase function</td>
</tr>
<tr>
<td>(\phi)</td>
<td>radiation direction azimuthal angle ((^\circ))</td>
</tr>
<tr>
<td>(\Theta)</td>
<td>scattering angle ((^\circ))</td>
</tr>
<tr>
<td>(\theta)</td>
<td>radiation direction polar angle ((^\circ))</td>
</tr>
<tr>
<td>(\tau)</td>
<td>optical thickness</td>
</tr>
<tr>
<td>(\omega)</td>
<td>single scattering albedo</td>
</tr>
</tbody>
</table>

Subscripts

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>blackbody</td>
</tr>
<tr>
<td>(HG)</td>
<td>Henyey–Greenstein</td>
</tr>
<tr>
<td>(i)</td>
<td>control-volume face</td>
</tr>
</tbody>
</table>


t radiation incident direction
l, l’ radiation directions
l’l from direction l’ into direction l

REFERENCES


heat transfer engineering


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