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Applicability of Phase-Function Normalization Techniques for Radiation Transfer Computation
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The applicability of recently developed four phase-function (PF) normalization techniques for modeling radiation transfer in strongly anisotropic scattering media is intensively examined using the discrete-ordinates method. The three simple techniques via normalization of only the forward- and/or backward-scattering directions have been shown to reduce normalization complexity while retaining diffuse radiation computation accuracy for Heney-Greenstein (HG) PFs. For Legendre PFs, however, such simple techniques are found to result in unphysical negative PF values at one or a few correction directions in some cases. Additionally, negative PF values can occur with these simple techniques for ballistic radiation transfer for both HG and Legendre PF types. If negative-intensity correction is applied, however, radiative heat transfer calculation can still converge regardless of the appearance of negative PF values. The relatively complex Hunter and Guo (2012) technique, in which normalization is realized through a correction matrix covering all discrete directions, is shown to be applicable for diffuse and ballistic radiation for both PF types.

1. INTRODUCTION

The discrete-ordinates method (DOM) is a popular numerical method for evaluating radiative heat transfer via solution of the equation of radiative transfer (ERT). First proposed as a method of determining astrophysical radiation [1], and later adopted as a method of solving the neutron-transport equation [2], the DOM was first applied as a method of solving thermal radiation in the 1980s. Fiveland [3] and Truelove [4] were pioneers in using the DOM as a method of determining steady-state radiative heat transfer. Later work by Mitra and Kumar [5] and by Guo and Kumar [6] extended the DOM as a means of solving the transient hyperbolic ERT in order to determine ultrafast laser radiation transfer in participating media. The main shortcomings to use of the DOM are the ray effects due to limited angular quadrature directions and numerical smearing due to spatial discretization [7, 8]. High-order numerical schemes have been considered to lessen the numerical smearing [9]. To reduce ray effects, vario
quadrature schemes have been implemented [10–12]. The DOM with unstructured grids has also been developed for irregular geometries [13].

The discretization of the continuous angular variation of radiation scattering into a finite set of discrete directions in the DOM has long been known to break the conservation of anisotropically scattered energy [14, 15], and only recently known to distort the phase-function (PF) asymmetry factor [16–18] for processes where scattering is highly anisotropic, as is the case for many practical scattering media such as biological tissue and packed beds. Additionally, for cases involving ballistic (collimated) radiation transport, ballistic out-scattered energy and asymmetry factor are also nonconserved [19] after discretization. PF normalization techniques have been introduced as a method to correct these nonconservations. Several different normalization methods have been proposed in the literature, with the majority of them able to conserve either scattered energy [14, 15, 20] or asymmetry factor [21], but not both. To this end, the present authors have developed a technique that is able to conserve scattered energy while simultaneously maintaining asymmetry factor for both Henyey-Greenstein (HG) [17] and Legendre PFs [19] via introducing a normalization matrix. This technique has been shown to lead to more accurate radiative transfer predictions in 3-D cubic enclosures as well [22].

Instead, it is noticed that recent normalization techniques developed by Mishchenko et al. [20] and Kamdem Tagne [21] tackled the issue of conserving either scattered energy or asymmetry factor in a much simpler manner. Rather than normalizing every value of the discrete PF, they wisely proposed normalization of solely the forward-scattering phase-function term and examined it in HG PFs. Such a treatment is computationally savvy, easy to implement, and retains the PF values for all but the correction direction. This idea, while simple, cannot conserve both

### NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A'$</td>
<td>forward-scattering normalization vector parameter</td>
</tr>
<tr>
<td>$A'^l$</td>
<td>backward-scattering normalization vector parameter</td>
</tr>
<tr>
<td>$B_l$</td>
<td>sum of discretized scattered energy, $= 1$ if conserved</td>
</tr>
<tr>
<td>$g$</td>
<td>asymmetry factor</td>
</tr>
<tr>
<td>$I$</td>
<td>radiative intensity, W/m$^2$sr</td>
</tr>
<tr>
<td>$M$</td>
<td>total number of discrete directions</td>
</tr>
<tr>
<td>$r$</td>
<td>position vector</td>
</tr>
<tr>
<td>$s$</td>
<td>unit direction vector</td>
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<tr>
<td>$w$</td>
<td>discrete direction weight</td>
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<td>$\Theta$</td>
<td>scattering angle, °</td>
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<tr>
<td>$\mu, \eta, \xi$</td>
<td>direction cosines</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>absorption coefficient, m$^{-1}$</td>
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<tr>
<td>$\sigma_s$</td>
<td>scattering coefficient, m$^{-1}$</td>
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</table>

| $\phi, \theta$ | radiation direction azimuthal angle and polar angle, ° |
| $\Phi$ | scattering phase function |
| $\Phi_0$ | normalized scattering phase function |
| $\omega$ | scattering albedo $= \sigma_a/$(\sigma_a + \sigma_s)$) |

<table>
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<th>Subscripts</th>
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<tr>
<td>$b$</td>
<td>blackbody</td>
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<td>$HG$</td>
<td>Henyey-Greenstein</td>
</tr>
<tr>
<td>$L$</td>
<td>Legendre</td>
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<tr>
<td>$N$</td>
<td>quadrature index</td>
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<td>$r$</td>
<td>radiation incident direction</td>
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<tr>
<td>$l, l'$</td>
<td>radiation directions</td>
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<tr>
<td>$l'/l$</td>
<td>from direction $l'$ to direction $l$</td>
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</table>
scattered energy and asymmetry factor quantities simultaneously, as normalization of only the forward term allows for just one constraint to be mathematically conserved. Expanding this idea, the present authors also developed a simple normalization technique [23], in which scattered energy and PF asymmetry factor are simultaneously conserved via normalization of both the forward- and backward-scattering directions for HG PFs. This allows for the simultaneous conservation of both scattered energy and asymmetry factor quantities, while reducing the necessity of the potentially computationally cumbersome normalization matrix.

Nevertheless, examination of the above-mentioned three simple techniques on Legendre polynomial PFs has not yet been performed and is necessary, as these functions are more representative of the highly oscillatory Mie scattering. Legendre PFs are not monotonic like the HG PFs, which have a distinct peak/minimal in the forward/backward direction. Normalization of only the forward and/or backward direction may not be sufficient or may lead to unphysical values for Legendre PFs. Additionally, the applicability of the simple techniques for PF normalization of ballistic radiation has not been addressed for either HG or Legendre PFs. Ballistic collimated radiation represents a wide range of important radiation transfer problems, e.g., in solar and laser radiation applications.

The purpose of this study is to examine and compare the applicability of the recent four PF normalization techniques for accurate radiative transfer computation in both diffuse and ballistic radiation situations with HG and Legendre PFs. Discretized scattered energy and asymmetry factor before and after application of the normalization techniques are investigated. Major issues, including the necessity of negative-intensity correction, are addressed with regard to specific normalization techniques. Comparisons of both diffuse and ballistic radiation transfer results generated with each normalization technique with higher-order finite-volume method (FVM) predictions are presented.

2. DISCRETIZATION OF THE ERT

Using general vector notation, the steady-state ERT for diffuse radiation intensity $I$ can be expressed as follows for a gray, absorbing-emitting, and anisotropically scattering medium:

$$\dot{s} \cdot \nabla I(r, \dot{s}) = - (\sigma_a + \sigma_s) I(r, \dot{s}) + \sigma_a I_b(r) + \frac{\sigma_s}{4\pi} \int \int \frac{I(r', \dot{s}') \Phi(\dot{s}', \dot{s})}{4\pi} d\Omega'$$  \hspace{1cm} (1)$$

where $\sigma_a$ and $\sigma_s$ are absorption and scattering coefficients, respectively. Using the 3-D Cartesian coordinate system, Eq. (1) can be expanded using the DOM into a simultaneous set of partial differential equations in discrete radiation directions $\dot{s}'$ as follows:

$$\mu' \frac{\partial I'}{\partial x} + \eta' \frac{\partial I'}{\partial y} + \zeta' \frac{\partial I'}{\partial z} = - (\sigma_a + \sigma_s) I' + \sigma_a I_b + S' \hspace{1cm} I = 1, 2, \ldots, M$$  \hspace{1cm} (2a)$$

$$S' = \frac{\sigma_s}{4\pi} \left( \sum_{j=1}^{M} w^j \Phi^j I^j + \sum_{B} \Phi^B I^B \right)$$  \hspace{1cm} (2b)$$
The continuous angular variation of radiative intensity is discretized into $M$ discrete radiation directions, with each direction $\vec{s}'$ defined by a polar angle $\theta$ and an azimuthal angle $\phi$, $\mu$, $\eta$, and $\xi$ are direction cosines corresponding to the $x$, $y$, and $z$ directions, respectively. The integral term in Eq. (1) is replaced by discrete quadrature summations in the source term of Eq. (2b), representing the in-scattering of both diffuse and ballistic radiation. For applications involving irradiation of collimated laser or solar incidence, it is crucial to include the ballistic term. In these summations, $w'_l$ is the DOM directional weighting factor corresponding to radiation direction $\vec{s}'_l$, $\Phi'^l$ is the scattering phase function for diffuse radiation between two arbitrary radiation directions $\vec{s}'$ and $\vec{s}'_l$, $I^B_l$ is the radiative heat flux of the ballistic radiation at a particular spatial location, and $\Phi^B_l$ is the scattering phase function for ballistic radiation between the direction of ballistic incidence $\vec{s}'^B_l$ and discrete direction $\vec{s}'_l$.

The Mie phase function $\Phi$, which describes radiation scattering in dielectric spheres, is highly oscillatory in nature and can be expressed through an infinite series of Legendre polynomials, as follows:

$$\Phi(\Theta) = 1 + \sum_{i=1}^{\infty} C_i P_i(\cos \Theta)$$

where the coefficients $C_i$ are determined via Mie theory, and $\Theta$ is the scattering angle between radiation directions $\vec{s}'$ and $\vec{s}'_l$. Numerical implementation of the Mie phase function is possible, but can be computationally cumbersome. To this end, it is common to approximate $\Phi(\Theta)$ by truncating the Legendre series to a finite number of terms, as follows:

$$\Phi_L(\Theta) = 1 + \sum_{i=1}^{N} C_i P_i(\cos \Theta)$$

where $N$ is the chosen term of approximation.

Another commonly implemented phase-function approximation is the Henyey-Greenstein (HG) phase function, whose analytical form is as follows:

$$\Phi_{HG}(\Theta) = \frac{1 - g^2}{(1 + g^2 - 2g\cos \Theta)^{1/2}}$$

where the phase-function asymmetry factor $g$ represents the averaged scattering direction cosine, which can be related to the Mie coefficient $C_1$ through the relation $g = C_1/3$.

In order to solve Eq. (2) using the DOM, the computational domain of interest is divided into control volumes, and spatial derivatives are approximated using the finite-volume approach. A DOM quadrature scheme defines the angular discretization and weighting factors of the discrete radiation directions. One of the more commonly implemented DOM quadratures is the level-symmetric $SN$ quadrature, where the subscript $N$ relates to the total number of discrete directions $M$ by the relation $M = N(N + 2)$. This traditional quadrature has a directional limit [10]. Other
quadrature sets, such as the $EO_N$ even–odd quadrature [11], $EQ_N$ equal weight quadrature [12], and $P_N–T_N$ Legendre-Chebyshev quadrature [10], have been developed as alternatives for discretizing the continuous angular variation. After the computational grid, quadrature scheme, and medium properties are set, Eq. (2) can be solved using a control-volume marching procedure. For brevity, further details on the DOM solution procedure are not repeated here, but are readily available in previous publications [17, 22].

3. NORMALIZATION METHODS

3.1. PF Normalization for Diffuse Radiation

It is widely recognized that the scattered energy should be accurately conserved after directional discretization, i.e.,

$$E = \frac{1}{4\pi} \sum_{l=1}^{M} \Phi^l w^l = 1 \quad l' = 1, 2, \ldots, M$$

(6)

While this condition is exactly conserved for isotropic scattering ($\Phi = 1$), it becomes increasingly violated as scattering anisotropy increases, resulting in substantial errors in radiative transfer computation of practical problems, in which scattering is always anisotropic.

In addition to scattered-energy conservation, recent awareness that the PF asymmetry factor $g$ should remain unchanged after directional discretization in order to retain prescribed medium properties [17] has grown, i.e.,

$$\frac{1}{4\pi} \sum_{l=1}^{M} \Phi^l w^l \cos \Theta^l \Theta^l= g \quad l' = 1, 2, \ldots, M$$

(7)

where $\Theta^l$ is the scattering angle between discrete directions $\hat{s}^l$ and $\hat{s}^l$.

PF normalization is a well-known method for ensuring conservation of scattered energy. In 2012, Hunter and Guo [17] published the first normalization technique that can satisfy both constraints. In this technique, the PF values are normalized as follows:

$$\Phi^l' = \left( 1 + A^l \right) \Phi^l$$

(8)

In the above, the normalization parameter matrix $A^l$ is determined such that $\Phi^l'$ satisfies Eqs. (6) and (7) simultaneously. DOM radiative transfer results generated using Hunter and Guo’s 2012 technique have been shown to conform accurately to both FVM and Monte Carlo (MC) predictions [22].

Mishchenko et al. [20], and later Kamdem Tagne [21], introduced simple PF normalization techniques through which either scattered-energy or asymmetry-factor conservation was achieved solely via normalization of the forward scattering term.
\[ \Phi^{e^f} = \left( 1 + A^f \right) \Phi^{e^f} \]  

(9)

where \( A^f \) is the forward-scattering normalization vector parameter, which can be expressed as follows for discrete direction \( s^f \):

\[ A^f = \left( 4\pi - \sum_{l=1}^{M} \Phi^{f l} w^l \right) / \left( \Phi^{e^f} w^f \right) \]  

(10a)

\[ A^f = \left( 4\pi g - \sum_{l=1}^{M} \Phi^{f l} w^l \cos \Theta^{f l} \right) / \left( \Phi^{e^f} w^f \right) \]  

(10b)

where Eq. (10a) lists the normalization parameters for Mishchenko et al.'s technique that will accurately conserve scattered energy, and Eq. (10b) lists the normalization parameters for Kamdem Tagne's technique that will accurately conserve asymmetry factor. While these methods (referred to as Mishchenko E and Kamdem Tagne g hereafter) are simple to implement, neither is able to accurately conserve both \( E \) and \( g \) simultaneously.

More recently, Hunter and Guo [23] developed a simple normalization technique for diffuse radiation of HG PFs, drawing on the approach used by both Mishchenko E and Kamdem Tagne g. Hunter and Guo proposed that, in order to conserve both scattered energy and asymmetry factor simultaneously, the backward-scattering term \( \Phi^{b^l} \) can be normalized in addition to the forward-scattering term \( \Phi^{e^f} \). Normalization of just these two terms allows for both of the critical constraints to be satisfied, while maintaining the majority of PF values. Applying this idea, the normalized values of the forward- and backward-scattering PF terms can be expressed as follows:

\[ \Phi^{e^f} = \left( 1 + A^f \right) \Phi^{e^f} \]  

(11a)

\[ \Phi^{b^l} = \left( 1 + B^{b^l} \right) \Phi^{b^l} \]  

(11b)

where \( A^f \) is the forward-scattering normalization vector parameter, \( B^{b^l} \) is the backward-scattering normalization vector parameter, and the superscript \( l^- \) refers to the direction directly opposite from \( s^f \), where \( \cos \Theta^{f l^-} = -1 \). This simple technique is able to accurately conserve both scattered energy and PF asymmetry factor simultaneously, without requiring manipulation of a potentially cumbersome normalization matrix. Since this technique is formally published in 2014, it is referred to as Hunter and Guo's 2014 technique hereafter.

### 3.2. PF Normalization for Ballistic Radiation

For cases involving ballistic radiation of collimated incidence, it has been previously shown [19] that special care must be given to the normalization of \( \Phi^{b^l} \) in order to ensure that ballistic radiation out-scattered energy and asymmetry factor...
are accurately conserved after directional discretization, in addition to normalization of the diffuse radiation counterpart. Except for the case where the direction of ballistic incidence \( \mathbf{s}^{bl} \) corresponds exactly to one of the discrete directions (in which the diffuse normalization parameters can be used), PF normalization of the ballistic radiation scattering is independent of diffuse radiation normalization.

Though the three simple techniques (Mishchenko E, Kamdem Tagne g, Hunter and Guo’s 2014) seem to be simpler than Hunter and Guo’s 2012 technique for diffuse radiation of HG PFs, their application to the ballistic radiation becomes more complicated. Due to the symmetry of quadrature, if the ballistic radiation direction does not correspond directly to one of the discrete directions, there will be no exactly forward- or backward-scattering \( \cos \Theta^{bl} = \pm 1 \) PF term to normalize, as directions with those cosines will not exist. Here it is proposed to normalize the PF for the directions where \( \cos \Theta^{bl} \) reaches its minimum and maximum. Applying this notion, the scattered-energy and asymmetry-factor conservation constraints for the PF of ballistic radiation for Hunter and Guo 2014 technique can be formulated as follows:

\[
E = \frac{1}{4\pi} \sum_{l=1}^{M} \Phi^{\beta_i} w' \cos \Theta^{bl} + \frac{N_{B^+}}{4\pi} \left( 1 + A^{B_+} \right) \Phi^{\beta_i B_+} w^{B_+} \cos \Theta^{B^+} \\
+ \frac{N_{B^-}}{4\pi} \left( 1 + B^{B_-} \right) \Phi^{\beta_i B_-} w^{B_-} \cos \Theta^{B_-} = 1
\]  

\[
\frac{1}{4\pi} \sum_{l=1}^{M} \Phi^{\beta_i} w' \cos \Theta^{bl} + \frac{N_{B^+}}{4\pi} \left( 1 + A^{B_+} \right) \Phi^{\beta_i B_+} w^{B_+} \cos \Theta^{B^+} \\
+ \frac{N_{B^-}}{4\pi} \left( 1 + B^{B_-} \right) \Phi^{\beta_i B_-} w^{B_-} \cos \Theta^{B_-} = g
\]

where \( A^{B_+} \) and \( B^{B_-} \) are the “forward” and “backward” normalization parameters. The directional superscripts \( B^+ \) and \( B^- \) represent the discrete directions that have the maximum and minimum scattering cosines as compared to the direction of collimated incidence, and the multiplicative factors \( N_{B^+} \) and \( N_{B^-} \) represent the number of discrete directions that attain such a maximum and minimum, respectively. Due to directional symmetry of quadrature, it is highly likely that multiple directions (1 to 6 directions in 3-D situations, 1 to 4 directions in 2-D situations) will share the same overall ballistic scattering cosine, and thus addition of these factors is necessary. Solving Eqs. (12a) and (12b) simultaneously for \( A^{B_+} \) and \( B^{B_-} \) yields the following expressions:

\[
A^{B_+} = \left[ 4\pi \left( \cos \Theta^{B_+} - g \right) + \sum_{l=1}^{M} \Phi^{\beta_i} w' \left( \cos \Theta^{bl} - \cos \Theta^{B^+_i} \right) \right] / \left( N_{B^+} \Phi^{\beta_i B_+} w^{B_+} \left( \cos \Theta^{B^-_i} - \cos \Theta^{B^+_i} \right) \right)
\]  

\[
B^{B_-} = \left[ 4\pi \left( \cos \Theta^{B_-} - g \right) + \sum_{l=1}^{M} \Phi^{\beta_i} w' \left( \cos \Theta^{bl} - \cos \Theta^{B^-} \right) \right] / \left( N_{B^-} \Phi^{\beta_i B_-} w^{B_-} \left( \cos \Theta^{B^+_i} - \cos \Theta^{B^-} \right) \right)
\]
\[ B^{B^+} = \left[ 4\pi \left( \cos \Theta^{B^+} - g \right) + \sum_{l=1}^{M} \Phi_l^{B^+} w_l \left( \cos \Theta_l^{B^+} - \cos \Theta_l^{B^-} \right) \right] / \left[ N^{B^-} \Phi_l^{B^-} w_l^{B^-} \left( \cos \Theta_l^{B^-} - \cos \Theta_l^{B^+} \right) \right] \]  

(13b)

Similar redefinition must be addressed for ballistic radiation PF normalization using both the Mishchenko E and Kamdem Tagne g techniques in the absence of backward-scattering term. Using the notion explained above, the ballistic radiation forward-scattering parameter \( A^{B^+} \) for Mishchenko E and Kamdem Tagne g, respectively, is

\[ A^{B^+} = \left( 4\pi - \sum_{l=1}^{M} \Phi_l^{B^+} w_l \right) / \left( N^{B^+} \Phi_l^{B^+} w_l^{B^+} \right) \]  

(14a)

\[ A^{B^+} = \left( 4\pi g - \sum_{l=1}^{M} \Phi_l^{B^+} w_l^\cos \Theta_l^{B^+} \right) / \left( N^{B^+} \Phi_l^{B^+} w_l^{B^+} \cos \Theta_l^{B^+} \right) \]  

(14b)

For Hunter and Guo’s 2012 technique, when the ballistic direction does not coincide with any discrete direction, it is straightforward to find a normalization vector, using least-squares or other methods, which obeys

\[ \Phi_l^{B^+} = \left( 1 + A^{B^+} \right) \Phi_l^{B^-} \quad l = 1, 2, \ldots, M \]  

(15a)

\[ \frac{1}{4\pi} \sum_{l=1}^{M} \Phi_l^{B^-} w_l = 1 \]  

(15b)

\[ \frac{1}{4\pi} \sum_{l=1}^{M} \Phi_l^{B^-} w_l^\cos \Theta_l^{B^+} = g \]  

(15c)

4. RESULTS AND DISCUSSION

As a means of examining the applicability of the four PF normalization techniques (Mishchenko E, Kamdem Tagne g, Hunter and Guo 2012 and 2014), the results and discussion are broken into two sections. First, a detailed examination of the impact of PF normalization for diffuse radiation is presented using Legendre polynomial PFs, as such functions are more general, and HG PFs have been previously examined. Values of the forward- and backward-scattering parameters that accurately conserve scattered energy and asymmetry factor are presented for varying DOM quadrature schemes and directional orders. Additionally, DOM radiative transfer profiles generated using the four normalization techniques are compared to high-order FVM predictions, as a means of gauging the accuracy of the normalized DOM. In the second section, the suitability of using the various normalizations to conserve ballistic radiation out-scattered energy and asymmetry factor and to
accurately predict ballistic radiation transport is investigated for both the HG and Legendre PFs.

The workstation used for computation is a Dell Optiplex 780, with an Intel Core 2 processor and 4.0 GB of RAM. DOM radiative transfer results were generated using the FORTRAN computing language. For all normalization techniques except Hunter and Guo’s 2012, all calculations were performed directly in the FORTRAN environment. For results generated using Hunter and Guo’s 2012 technique, the normalization matrix $A^{\text{fit}}$ that satisfies the conservation constraints was solved using MATLAB’s built-in least-squares approximation solver for simplicity, and then imported back into FORTRAN for radiative transfer computation. Generation of the normalization matrix takes less than 1 min CPU time. The CPU times and memories used for radiative transfer computation for the four normalization techniques are quite similar.

### 4.1. Diffuse Radiation with Legendre PFs

Table 1 presents the lack of either scattered-energy or asymmetry-factor conservation after directional discretization for the two normalization techniques that do not conserve both quantities simultaneously. The three prescribed $g$ values correspond to three Legendre PFs available in the literature: Mie coefficients $C_i$ for the 9-term $g = 0.6697$ phase function were presented by Kim and Lee [24], while coefficients for the 26-term $g = 0.8189$ and 27-term $g = 0.9273$ were presented by Lee and Buckius [25]. $E$ and $g$ values after discretization are tabulated for the DOM $EO_N$ quadrature, with indices $N = 4, 6, 8, 10, 12, \text{ and } 16$, corresponding to $M = 24, 48, 72$.

**Table 1.** Examination of the discretized scattered energy and asymmetry factor for diffuse radiation using the $EO_N$ quadrature for Legendre PFs with and without normalization

<table>
<thead>
<tr>
<th>Pres. $g$</th>
<th>$N$</th>
<th>Discretized $E$</th>
<th>Discretized $g$</th>
<th>$%$ Diff. in $(1-g)$</th>
<th>Discretized $E$</th>
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<td>0.9998</td>
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80, 120, 168, and 288 discrete directions, respectively. Values of $E$ and $g$ are similar for $S_N$, $P_3 - T_N$, and $E_Q$, and thus only one quadrature is shown, for brevity. The values without normalization are shown as a means of validating the necessity of normalization. For the Mishchenko $E$ technique, only discretized $g$ values are presented, since $E$ is always conserved after normalization. Conversely, only the $E$ value is tabulated for the Kamdem Tagne $g$ technique, because its $g$ is conserved. Additionally, percent differences in both $E$ and $(1 - g)$ are also listed. Change in scattering effect is manifested in the difference in reduced scattering coefficient, or say, $(1 - g)$ according to scaling law [25, 26], and not $g$ itself, and thus these values are presented in order to more properly gauge the impact of improper conservation. Hunter and Guo’s 2012 and 2014 techniques accurately conserve both $E$ and $g$ quantities, so there is no need to list them in the table.

For the 9-term $g = 0.6697$ PF, discrepancies in both scattered energy and asymmetry factor after directional discretization are minimal prior to normalization except for the lowest-order quadrature ($N = 4$). However, significant discrepancies are witnessed for both the 26-term $g = 0.8189$ and 27-term $g = 0.9273$ PFs even with high-order quadratures. It is interesting to note that, for all directional orders, the breakdown of conservation for $g = 0.8189$ is worse than for $g = 0.9273$. This counters findings for the HG PF [23], where scattered-energy and asymmetry-factor conservation became increasingly worse as $g$ increased, because HG PF irregularity depends on $g$ only, while the oscillations in the Legendre polynomial PFs depend not only on asymmetry factor but also on the number of higher-order terms in the expansion and their corresponding Mie coefficients. Application of the Mishchenko $E$ technique also improves the preservation of asymmetry factor. However, the change in scattering effect can be large, reaching 21% for $g = 0.9273$ with $N = 6$. Implementation of the Kamdem Tagne $g$ technique does not conserve scattered energy. Slight changes in scattered-energy conservation can drastically alter radiative transfer predictions.

When the three simple normalization techniques (Mishchenko $E$, Kamdem Tagne $g$, Hunter and Guo’s 2014) were applied to the HG PFs, corrected $\Phi$ values were positive for all $g$ values [23]. For the general Legendre polynomial PFs, however, we found that application of the simple techniques in some cases can lead to either forward- or backward-scattering normalization parameters less than $-1$, which lead to physically unrealistic negative values in the normalized PF. This finding is new. Table 2 lists all the quadrature schemes and directional orders for the three Legendre PFs where backward normalization parameters less than $-1$ occur for Hunter and Guo 2014 technique. For nonlisted quadrature schemes and orders, we did not find backward parameters $<-1$.

Figures 1a and 1b plot discretized Legendre PF values versus scattering-angle cosine generated with the four normalization techniques for two cases: the $EO_8$ quadrature with $g = 0.8189$, and the $EO_{16}$ quadrature with $g = 0.9273$. Theoretical values are plotted for comparison. All four techniques produce similar PF values at most locations, except at the forward- and backward-directions $\cos \Theta = \pm 1$. The inset in Figure 1a shows that the three simple normalization techniques result in negative PF values in the forward direction, indicating that the simple idea of solely normalizing the forward/backward term(s) might be unsuitable for certain Legendre PFs, though it was shown to be acceptable for HG PFs [20, 21, 23]. In Figure 1b and
its inset, large differences between the discretized values generated with Hunter and Guo’s 2014 technique and the theoretical are seen at \( \cos \Theta = -1 \), with multiple backward directions altered to a negative PF value.

In Figure 2, the minimum values of forward-scattering parameter (or \( A^f \) for Hunter and Guo’s 2012 technique) are plotted for the principal octant directions using the \( EO_8 \) quadrature with \( g = 0.8189 \) for the four normalization techniques. For all 10 principal octant directions, forward parameters for Mishchenko E, Kamdem Tagn\( e \) g, and Hunter and Guo’s 2014 techniques are nearly identical, with parameters of less than \(-1\) occurring for three directions (negative PF values). For all but those three directions, Hunter and Guo’s 2012 technique produces nearly identical forward parameters as the other three techniques. As seen in Figures 1 and 2, Hunter and Guo’s 2012 technique is able to avoid the critical issue of negative PF values. Forward-scattering parameters \(<-1\) were not seen for any other combination of quadrature or PF type examined in this study.

The appearance of negative PF values in the three simple techniques is potentially a critical issue for accurate radiation transfer computation. In order to investigate said impact, a benchmark test problem involving steady-state radiation transfer in a cubic enclosure of edge length \( L \) is examined. The cubic enclosure houses a purely scattering medium \((\Theta = 1.0)\) to highlight the influence of anisotropic scattering. The staggered control-volume spatial grid was taken as \((N_x \times N_y \times N_z) = 27 \times 27 \times 27\) with a uniform grid size \(1/25\). The spatial coordinates are nondimensionalized in the following manner: \( x^* = x/L, y^* = y/L, \) and \( z^* = z/L \). The medium and enclosure walls are taken to be cold and black, except for the wall at \( z^* = 0\), which is taken as a diffuse emitter with unity emissive power.

It was found during simulation that negative PF values resulted in negative intensity values. Negative intensities tend to occur early in the simulation for a cold medium (where initial intensity is 0), and disappear as radiant energy spreads through the medium. This issue does not occur for every case where forward or backward parameters result in negative PF values, as factor such as optical properties and the value of the DOM quadrature weighting factor play a large role in the intensity value generated from the ERT. However, in cases where intensities become negative, the solution diverges at the next iteration, and a convergent ERT solution

<table>
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<th>( g )</th>
<th>Minimum backward parameters</th>
</tr>
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</tr>
<tr>
<td></td>
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<td>(-8.104)</td>
</tr>
<tr>
<td>( S_N )</td>
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<td>0.6697</td>
<td>(-1.259)</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>0.9273</td>
<td>(-22.71)</td>
</tr>
<tr>
<td></td>
<td>16</td>
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<td>(-3.954)</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>0.8189</td>
<td>(-3.035)</td>
</tr>
<tr>
<td></td>
<td>16</td>
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<td>(-15.99)</td>
</tr>
<tr>
<td>( EO_N )</td>
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<td>(-1.063)</td>
</tr>
<tr>
<td></td>
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<td>(-13.49)</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>0.9273</td>
<td>(-304.5)</td>
</tr>
<tr>
<td>( EQ_N )</td>
<td>16</td>
<td>0.9273</td>
<td>(-3.911)</td>
</tr>
</tbody>
</table>
cannot be obtained. As a means of correcting this issue, the positive scheme [2,6] for negative-intensity correction was implemented, in which any negative intensities are set to zero.

Figures 3a and 3b compare nondimensional heat flux distributions calculated using the four normalization techniques. The optical thickness of the medium is $\tau = (\sigma_a + \sigma_s)L = 10.0$. Results are generated using the $EO_8$ quadrature and $g = 0.8189$ in Figure 3a, and with the $EO_{16}$ quadrature and $g = 0.9273$ in Figure 3b. Heat flux profiles determined using the FVM are also presented for comparison. For the FVM calculations, we used the $FT_N$-FVM quadrature set of
Kim and Huh [27] with a large number of discrete directions ($M = 840$), in order to conserve scattered energy within $10^{-4}\%$ and asymmetry factor within 0.25% for both PFs without normalization.

For the results generated in Figure 3a, negative intensities were encountered for the three simple normalization techniques, corresponding to the negative PF values shown in Figure 1a. As such, the negative-intensity correction was applied. Hunter and Guo’s 2012 technique is taken as a validation basis for DOM results, as this negative issue does not occur. Heat fluxes generated using Hunter and Guo’s 2014 technique with negative-intensity correction conform accurately to those generated using Hunter and Guo’s 2012 technique, with differences of less than 1.5% seen at $y^* = 0.1$, and less than 1% at $y^* = 0.5$. Profiles generated using the Mishchenko E technique overpredict Hunter and Guo’s 2012 technique by about 3% at both locations, corresponding to the fact that asymmetry factor is not accurately conserved (discretized $g = 0.8233$). Conversely, a lack of scattered-energy conservation by the Kamdem Tagne g technique (discretized $E = 0.9957$) results in heat-flux underprediction of around 5% at both locations as compared with Hunter and Guo’s 2012 technique. When compared with higher-order FVM heat flux, the four techniques produce heat fluxes that differ by a maximum of 8.8% at $y^* = 0.1$ and 5.4% at $y^* = 0.5$. These are acceptable differences, considering that only $M = 80$ directions were used for the DOM results.

Due to the large absolute values of the backward parameters in Hunter and Guo’s 2014 technique, it is of interest to show normalized heat fluxes at the hot wall, or $Q(x^*, y^*, z^* = 0.0)$, in order to see if these parameters result in a significant change in heat flux getting back into the hot wall. From Figure 3b, it is seen that results generated using Hunter and Guo’s 2014 technique conform to Hunter and Guo’s 2012 technique and to higher-order FVM within 0.15% for all $x^*$ and $y^*$ locations examined. Heat fluxes generated with the other two normalization techniques differ by less than 0.5% at maximum.
For applications involving turbid media, such as light transport through biological tissue, the reduced optical thickness is commonly much greater than 1. Figure 4 presents an analysis of the impact of PF normalization on heat flux in an optically thick medium ($\tau = 100.0$). The results are generated with the $EO_{16}$ quadrature with $g = 0.9273$. While DOM profiles generated using the other three techniques conform to within 2% of each other and within 7% to higher-order FVM, the slight lack of energy conservation ($E = 0.9998$) inherent in the Kamdem Tagne g technique results in drastic errors in heat flux profile of up to 18% as compared with Hunter

![Figure 3](image-url)
and Guo’s 2012 technique. Analysis of the optically thick case indicates the absolute necessity of conserving scattered energy after discretization. Conversely, small discrepancies in asymmetry factor inherent in the Mishchenko E technique \((g = 0.9276)\) have only a minimal impact on radiative transfer results as compared with techniques that conserve both quantities.

The results in Figures 3 and 4 show that even though the three simple techniques can result in negative Legendre PF values in some cases, diffuse radiation transfer predictions are largely unaffected as long as a negative-intensity correction is put in place. It should also be mentioned that the results generated with other quadrature sets \((SN, PN \sim TN, EQN)\) are similar to the \(EON\) quadrature results.

### 4.2. Ballistic Radiation with HG and Legendre PFs

In order to analyze the impact of PF normalizations on ballistic radiation, a benchmark problem similar to that analyzed in the previous section is examined. All simulation conditions are identical to the problem addressed for the diffuse-radiation case, except for the boundary condition at \(z^* = 0\), where the wall is taken to be cold and is irradiated by a collimated normal incidence of unity intensity. It should be mentioned that, for all forthcoming radiation transfer results, the phase function for diffuse component scattered away from the collimated radiation has been normalized using the same technique as for the ballistic component.

Table 3 lists discretized scattered energy and asymmetry factor values prior to PF normalization of ballistic radiation for various DOM quadrature schemes and directional orders. The data are tabulated for the HG PFs with \(g = 0.8000\) and 0.9300, as well as for the Legendre PF with \(g = 0.8189\). Results are presented for four DOM quadratures: \(S_N, EON, P_N \sim T_N\) and \(EQN\) equal-weight quadrature. The data in

![Figure 4. Comparison of heat flux profiles generated with FVM and various DOM diffuse-radiation normalization techniques in an optically thick medium.](image)
Table 3 are valid for a normally incident ballistic irradiation, as Hunter and Guo previously showed that changes in incident angle dramatically impact ballistic-radiation scattered energy and asymmetry-factor conservation values [19]. For $g = 0.8000$, significant errors in discretized scattered energy and asymmetry factor are seen for all quadratures with $N \leq 12$. At $N = 16$ for $S_N$, the difference in scattering effect ($1 - g$) is still over 18%. As $g$ is increased to 0.9300, the breakdown of conservation for both quantities becomes much worse. Only the $EO_{16}$ quadrature reasonably preserves $E$ and $g$. For the Legendre $g = 0.8189$ phase function, $E$ and $g$ are accurately conserved to less than 0.001% for the $P_{16} - T_{16}$ quadrature, while errors are significant for all other quadratures and discrete-direction numbers. Thus, normalization for ballistic radiation is more critical than for diffuse radiation.

Table 4 lists discretized values of normally incident ballistic radiation out-scattered energy and asymmetry factor with application of the Mishchenko $E$ and Kamdem Tagné $g$ techniques to $U_{ulBl}$ (Hunter and Guo’s 2012 and 2014 techniques conserve both quantities and are not shown) for HG $g = 0.9300$ and Legendre $g = 0.8189$. When the Mishchenko $E$ technique is applied, large deviations in asymmetry factor are seen for low-order quadrature for both PFs. Even at higher-order quadratures ($N \geq 12$), change in scattering-effect ranges can be over 10%. Conversely, for the Kamdem Tagné $g$ technique, deviations in scattered energy of less than 1% are consistently seen only for $N \geq 12$.
In Figure 5a, the forward- and backward-scattering normalization parameters for ballistic radiation generated using Hunter and Guo’s 2014 technique are presented for varying DOM quadratures and directional orders. Parameters are presented for HG $g = 0.9300$. The HG phase function is chosen to simplify the analysis, as the HG function depends solely on asymmetry factor, and both Hunter and Guo’s 2012 and 2014 techniques are suitable for HG PFs of diffuse radiation. For all directional orders, the forward-scattering normalization parameters are positive. Although not presented here, for brevity, ballistic-radiation normalization using the other three normalization techniques resulted in similar forward-scattering normalization parameters. Examination of the backward-scattering normalization parameters in Figure 5a reveals a critical issue: $\Phi_0 < -1$ for nearly all the tested quadrature schemes and discrete direction numbers, which will result in negative values of $\Phi_{1\beta}$. 

Figure 5b compares the backward-scattering normalization parameters generated with Hunter and Guo’s 2014 ballistic technique to parameters generated with Hunter and Guo’s 2012 ballistic technique for the same discrete directions. Parameters generated using Hunter and Guo’s 2014 technique vary greatly with quadrature scheme, and parameters $\Phi_0 < -1$ are common, even with $N = 16$ for $S_N$, $E_{ON}$, and $P_{N-T_N}$, while for Hunter and Guo’s 2012 technique, parameters less than $-1$ occur only for the lowest quadrature order ($N = 4$). Alteration of every value of $\Phi_{1\beta}$ in Hunter and Guo’s 2012 technique allows for less drastic alteration of the backward-scattering terms, indicating that it is a more suitable technique.

Figures 6a and 6b present heat flux generated using the DOM $P_{N-T_N}$ quadrature with various ballistic radiation normalization techniques. FVM results with $M = 840$ are also presented for comparison. The method used for ballistic-radiation normalization does not have a large impact on DOM radiative transfer results. For $M = 80$ and 288, the percent differences among the four ballistic techniques are less than 2.5% and 0.5%, respectively. Directional order does have an impact, however, when investigating a comparison to higher-order FVM. For $M = 80$, significant

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<th>$N$</th>
<th>$P_{N-T_N}$</th>
<th>$S_N$</th>
<th>$E_{ON}$</th>
<th>$E_{QN}$</th>
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In Figure 5a, the forward- and backward-scattering normalization parameters for ballistic radiation generated using Hunter and Guo’s 2014 technique are presented for varying DOM quadratures and directional orders. Parameters are presented for HG $g = 0.9300$. The HG phase function is chosen to simplify the analysis, as the HG function depends solely on asymmetry factor, and both Hunter and Guo’s 2012 and 2014 techniques are suitable for HG PFs of diffuse radiation. For all directional orders, the forward-scattering normalization parameters are positive. Although not presented here, for brevity, ballistic-radiation normalization using the other three normalization techniques resulted in similar forward-scattering normalization parameters. Examination of the backward-scattering normalization parameters in Figure 5a reveals a critical issue: $\Phi_0 < -1$ for nearly all the tested quadrature schemes and discrete direction numbers, which will result in negative values of $\Phi_{1\beta}$.

Figure 5b compares the backward-scattering normalization parameters generated with Hunter and Guo’s 2014 ballistic technique to parameters generated with Hunter and Guo’s 2012 ballistic technique for the same discrete directions. Parameters generated using Hunter and Guo’s 2014 technique vary greatly with quadrature scheme, and parameters $\Phi_0 < -1$ are common, even with $N = 16$ for $S_N$, $E_{ON}$, and $P_{N-T_N}$, while for Hunter and Guo’s 2012 technique, parameters less than $-1$ occur only for the lowest quadrature order ($N = 4$). Alteration of every value of $\Phi_{1\beta}$ in Hunter and Guo’s 2012 technique allows for less drastic alteration of the backward-scattering terms, indicating that it is a more suitable technique.

Figures 6a and 6b present heat flux generated using the DOM $P_{N-T_N}$ quadrature with various ballistic radiation normalization techniques. FVM results with $M = 840$ are also presented for comparison. The method used for ballistic-radiation normalization does not have a large impact on DOM radiative transfer results. For $M = 80$ and 288, the percent differences among the four ballistic techniques are less than 2.5% and 0.5%, respectively. Directional order does have an impact, however, when investigating a comparison to higher-order FVM. For $M = 80$, significant
differences of up to 25% are seen between normalized DOM and FVM profiles, while these discrepancies reduce to less than 2% with increase to $M = 288$.

Figures 7a and 7b present heat flux for an optically thick medium with $\tau = 100.0$. All other conditions are the same as for Figure 6. Hunter and Guo’s 2012 and 2014 normalizations result in nearly identical DOM heat flux profiles that conform accurately within 2% to high-order FVM. However, results generated using Mishchenko E and Kamdem Tagne g exhibit stronger discrepancies. Lack of

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{(a) Forward and backward normalization parameters for ballistic radiation using Hunter and Guo’s 2014 technique and (b) comparison of backward-scattering normalization parameters between Hunter and Guo’s 2012 and 2014 techniques for ballistic radiation for HG PF.}
\end{figure}
asymmetry-factor conservation in Mishchenko E leads to overpredictions in heat flux. Conversely, lack of energy conservation in Kamdem Tagne g results in extreme underpredictions, illustrating the crucial importance of energy conservation for applications involving optically thick media.

Figures 8a and 8b plot heat flux generated using $EO_8$ ($M=80$) and $EO_{16}$ ($M=288$) as the representative quadrature sets and the Legendre $g=0.8189$. All other properties are identical to those in Figure 6. For $M=80$, heat fluxes generated using Hunter and Guo’s 2012 and 2014 techniques differ from each other by less
than 1% at all locations, while over- and underpredictions of $\sim2–3\%$ are witnessed for Mishchenko E and Kamdem Tagne, respectively. The differences between DOM and higher-order FVM for $M = 80$ are fairly significant due to ray effect, with all techniques overpredicting FVM by between 4% and 20%. When direction number is increased to $M = 288$, heat fluxes generated using the four ballistic radiation normalization techniques conform accurately to one another within 0.2% at all locations. Additionally, ray effect is effectively mitigated, with percentage differences of less than 4% at maximum seen between DOM and high-order FVM.

Figure 7. Comparison of heat flux profiles generated with FVM and various DOM ballistic radiation normalization techniques in an optically thick medium.
The issue of negative PF function values after normalization was also found for ballistic radiation. As long as the negative-intensity correction is enforced, its impact on radiation transfer is minimal. The choice of ballistic-normalization technique does not appear to drastically impact radiative transfer results for either the HG or Legendre PF, except for cases where the medium is optically thick, in which conservation of asymmetry factor and especially scattered energy is crucial and Mishchenko E and Kamdem Tagne g fail either. Though only representative quadrature results were presented, we examined the calculations using DOM $S_N$, $EO_N$, 

Figure 8. Comparison of heat flux profiles generated using FVM and DOM with various ballistic-radiation normalization techniques for Legendre PF.
$P_{N-T_N}$ and $EQ_N$ quadratures and did not find big difference among the four different quadratures.

Finally, it should be mentioned that, for the case where the direction of ballistic incidence matches one of the predetermined quadrature directions, the ballistic normalization parameter (s) will be identical to the diffuse normalization parameter(s) for that quadrature direction. In such cases, if diffuse forward parameters of $<-1$ occur, then the ballistic forward parameters will also be $<-1$, resulting in negative phase-function values and mandating the necessity of negative-intensity correction. Additionally, for ballistic incident directions that are both non-normal and non-aligned with a predetermined quadrature direction, negative forward parameters are encountered for certain Legendre PFs. For example, for the $EO_8$ quadrature with the $g=0.8189$ Legendre PF, a forward ballistic-normalization parameter of $-1.02$ occurs for ballistic incidence at polar angle $43^\circ$ and azimuthal angle $5^\circ$ after application of Hunter and Guo’s 2014 ballistic normalization. Thus, for general Legendre PFs, the existence of negative PF values and the necessity of negative-intensity correction associated with the three simple normalization techniques should always be borne in mind.

5. CONCLUSIONS

For the first time, PF normalization for the three simple forward/backward parameters normalization techniques is formulated for ballistic radiation and their applicability to general Legendre PFs for both diffuse and ballistic radiation transfer is examined and compared with Hunter and Guo’s 2012 normalization matrix technique. The following important guidelines are summarized.

1. Hunter and Guo’s 2012 technique is applicable for both HG and Legendre PFs, as it never results in negative PF values for diffuse radiation for all quadrature sets and directional orders, as well as for ballistic radiation when directional order $N \geq 6$. This technique conserves both scattered energy and asymmetry factor simultaneously for both diffuse and ballistic radiation, resulting in accurate radiative transfer predictions as compared with FVM and Monte Carlo [22].

2. Hunter and Guo’s 2014 technique is highly applicable for diffuse radiation with HG PFs. For Legendre PFs, however, this technique may result in negative PF values for some quadrature schemes and directional orders, because Legendre PFs are not monotonic like the HG PFs, which have a distinct peak/minimal in the forward/backward direction. For ballistic radiation, this technique may result in negative PF values regardless of PF type.

3. Both Mishchenko et al.’s and Kamdem Tagne’s techniques are prone to negative forward PF values for diffuse and ballistic radiation with Legendre PFs for some quadrature schemes and directional orders, but no such issue occurs for the HG PFs. However, lack of either scattered-energy or asymmetry-factor conservation inherent in these two techniques can cause significant errors in radiative transfer predictions, especially for optically thick media.

4. The occurrence of negative PF values after normalization appears only in one or a few correction directions. By application of negative-intensity correction, converged radiation transfer results can still be obtained, and the impact of the
negative value issue after correction is slight on the overall radiation heat transfer calculations examined.

5. The three simple normalization techniques alter only the forward/backward-scattering term(s) and introduce one/two normalization vector(s), allowing for retention of the majority of PF values and shape. However, the magnitude of the change is sometimes dramatic. Although Hunter and Guo’s 2012 technique uses a normalization matrix for diffuse radiation or a normalization vector for ballistic component and alters all the PF terms, the alterations are small in magnitude (generally smaller than the oscillatory magnitude of Mie scattering); and thus, the PF shape is still well preserved.

6. In terms of computational efficiency on radiation transfer, there are no obvious differences among the four different normalization techniques, because the normalization matrix/vector in Hunter and Guo 2012 technique can be easily pre-generated and input conveniently as a built-in quantity.

REFERENCES