COMPARISON OF PHASE FUNCTION NORMALIZATION TECHNIQUES FOR RADIATIVE TRANSFER ANALYSIS USING DOM

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ABSTRACT
Five phase-function (PF) normalization techniques are compared using the discrete-ordinate method (DOM) for modeling diffuse radiation heat transfer in participating media. Both the mathematical formulation and the impact on the conservation of both scattered energy and PF asymmetry factor for both Henyey-Greenstein (HG) and Legendre PF distributions are presented for each technique. DOM radiation transfer predictions generated using the five normalization techniques are compared to high-order finite-volume method, to gauge their accuracy. The commonly implemented scattered energy averaging technique cannot correct asymmetry factor distortion after angular discretization, and thus large errors due to angular false scattering are prevalent. Another three simple techniques via correction of one or two terms in the PF are shown to reduce normalization complexity whilst retaining diffuse radiation computation accuracy for HG PFs. However, for Legendre PFs, such simple normalization is found to result in unphysical negative PF values at one or few correction directions. The relatively complex Hunter and Guo 2012 technique, in which normalization is realized through a correction matrix covering all discrete directions, is shown to be highly applicable for both PF types.

INTRODUCTION
Originally proposed as a method of determining astrophysical radiation [1], and later adopted as a method of solving the neutron-transport equation [2], the Discrete-Ordinates Method (DOM) has become a popular numerical method for evaluating radiation heat transfer via solution of the Equation of Radiation Transfer (ERT). Use of the DOM for determining steady-state radiation heat transfer was pioneered in the 1980’s [3,4]. In the following two decades, an important extension of the DOM to solve the transient hyperbolic ERT was proposed [5,6], in order to accurately determine ultrafast radiation transfer in participating media. While popular and easy to implement, the DOM suffers from two major numerical shortcomings: numerical smearing error due to spatial discretization, and ray effect error due to angular discretization [7,8]. High-order numerical schemes were considered to mitigate numerical smearing error [8]. Reduction of ray effect error has been achieved via use of different quadrature schemes [9-11]. Additionally, DOM with unstructured grids has also been developed for use with irregular geometries [12].

It is well known that the conservation of anisotropically scattered energy is broken after discretization of the continuous angular variation of radiation scattering into a finite set of discrete directions using the DOM [13, 14]. However, it has recently become clear that angular discretization additionally distorts the phase-function (PF) asymmetry factor [15-17] for processes where scattering is highly anisotropic, which can include many practical scattering media such as packed beds and biological tissues. PF normalization techniques become a popular method to correct these non-conservations. Multiple normalization methods have been proposed in literature, with the majority of them able to conserve either scattered energy [13, 14, 19] or asymmetry factor [20], but not both. Hunter and Guo developed a technique that can simultaneously conserve both quantities for both Henyey-Greenstein (HG) [16] and Legendre PFs [18] through use of a normalization matrix, allowing for more accurate radiation transfer predictions in 3-D cubic enclosures [21].

Other recent normalization techniques developed by Mishchenko et al. [19] and Kamdem Tagne [20] wisely tackled the issue of conserving either scattered energy or asymmetry factor in a simple manner. Rather than normalizing every value of the discrete PF, as had been done in many previous techniques, they proposed normalization of solely the forward-scattering phase-function term for HG PFs. Such PF treatment retains the PF value for most discrete direction combinations
while being computationally savvy and easy to implement. This idea, while simple, cannot simultaneously conserve energy and asymmetry factor. Expanding on this idea, Hunter and Guo also developed a simple normalization technique [22], in which scattered energy and PF asymmetry factor are simultaneously conserved via normalization of both the forward- and backward-scattering directions for HG PFs. While several normalization techniques have been presented through the years, a detailed comparison of the impact of said techniques has not been published.

The purpose of this study, therefore, is to examine and compare the applicability of five PF normalization techniques for determining accurate radiation transfer with both HG and Legendre PFs. Discretized scattered energy and asymmetry factor before and after application of the normalization techniques are investigated. Major issues, including the necessity of negative intensity correction, are addressed with regards to specific normalization techniques. Comparisons of radiation transfer results generated with each normalization technique with higher-order finite volume method (FVM) predictions are presented. Finally, conclusions as to the applicability of each technique are outlined.

**NOMENCLATURE**

- \( A^{f_i}, B^{r_i} \): Forward- and backward-scattering normalization vector parameters
- \( A^{t_i} \): Normalization coefficients
- \( g \): Asymmetry factor
- \( I \): Radiative intensity (W/m²sr)
- \( M \): Total number of directions
- \( r \): Position vector
- \( \hat{s} \): Unit direction vector
- \( w \): Discrete direction weight

**Greek Symbols**

- \( \sigma_a \): Absorption coefficient (m⁻¹)
- \( \sigma_s \): Scattering coefficient (m⁻¹)
- \( \mu, \eta, \xi \): Direction cosines
- \( \Phi \): Scattering phase function
- \( \Phi^t \): Normalized scattering phase function
- \( \theta, \phi \): Radiation direction polar and azimuthal angle (°)
- \( \omega \): Scattering albedo, \( = \sigma_s/ (\sigma_a + \sigma_s) \)

**Subscripts**

- \( b \): Blackbody
- \( HG \): Henyey-Greenstein
- \( L \): Legendre
- \( N \): DOM Quadrature index

**Superscripts**

- \( ' \): Radiation incident direction
- \( l, l' \): Radiation directions
- \( l,l' \): From direction \( l' \) into direction \( l \)

**DISCRETIZATION OF ERT**

The steady-state ERT of diffuse radiation intensity \( I \) can be expressed, in general vector notation, as follows for a gray, absorbing-emitting, and anisotropically scattering medium:

\[
\hat{s} \cdot \nabla I(r, \hat{s}) = -\left( \sigma_a + \sigma_s \right) I(r, \hat{s}) + \sigma_s l_b(r)
\]

\[+
\frac{\sigma_s}{4\pi} \int I(r, \hat{s}') \Phi(\hat{s}', \hat{s}) d\Omega
\]

(1)

where \( \sigma_a \) and \( \sigma_s \) are medium absorption and scattering coefficients, respectively. Using the 3-D Cartesian coordinate system, Eq. (1) can be expanded into a simultaneous set of partial differential equations in discrete radiation directions \( \hat{s}^l \) using the DOM, as follows:

\[
\mu^l \frac{\partial I^l}{\partial x} + \eta^l \frac{\partial I^l}{\partial y} + \xi^l \frac{\partial I^l}{\partial z} = -(\sigma_a + \sigma_s) I^l + S^l,
\]

\[l = 1, 2, ..., M\]

(2a)

\[
S^l = \sigma_s l_b + \frac{\sigma_s}{4\pi} \sum_{i=1}^{M} w^i \Phi^{t'l} I^{t' l^i}
\]

(2b)

The continuous angular variation of radiative intensity is discretized into \( M \) discrete radiation directions, with each direction \( \hat{s}^l \) defined by both polar angle \( \theta \) and azimuthal angle \( \phi \). The direction cosines \( \mu, \eta, \xi \) correspond to the \( x-, y-, \) and \( z- \) directions, respectively. The integral term in Eq. (1), which represents the in-scattering of diffuse radiation, is approximated in the source term of Eq. (2b) as a discrete quadrature summation. In said summation, \( w^t'l \) is the DOM directional weighting factor corresponding to radiation direction \( \hat{s}^l \), and \( \Phi^{t'l} \) is the scattering phase-function for diffuse radiation between two arbitrary radiation directions \( \hat{s}^l \) and \( \hat{s}^l' \).

The Mie scattering phase function \( \Phi \), which is valid for radiation scattering in dielectric spheres, is highly-oscillatory in nature and can be expressed through an infinite series of Legendre polynomials, as follows:

\[
\Phi(\Theta) = 1 + \sum_{l=1}^{\infty} C_l P_l(\cos \Theta)
\]

(3)

where \( \Theta \) is the scattering angle between radiation directions \( \hat{s}^l \) and \( \hat{s}^l' \), and the coefficients \( C_l \) are determined via Mie theory. Numerical implementation of the Mie phase function can be computationally cumbersome, and thus it is common to approximate \( \Phi(\Theta) \) by truncating the Legendre series to a finite number of terms:

\[
\Phi_L(\Theta) = 1 + \sum_{l=1}^{N} C_l P_l(\cos \Theta)
\]

(4)

where \( N \) is the chosen term of approximation.
Another commonly implemented phase-function approximation is the Heney-Greenstein (HG) phase function, whose analytical form is as follows:

\[
\Phi_{HG}(\Theta) = \frac{1 - g^2}{1 + g^2 - 2g \cos \Theta}^{1/2}
\]

where the phase-function asymmetry factor \( g \) represents the averaged scattering direction cosine, which can be related to the Mie coefficient \( C_1 \) through the relation \( g = C_1/3 \).

In order to solve Eq. (2) using the DOM, the computational domain of interest is divided into control volumes, and spatial derivatives are approximated using the finite volume approach. A DOM quadrature scheme defines the angular discretization and weighting factors of the discrete radiation directions. One commonly implemented DOM quadrature is the level-symmetric \( S_N \) quadrature, where \( N \) relates to the total number of discrete directions \( M \) by the relation \( M = N(N + 2) \). This traditional quadrature has a directional limit [9]. Other quadrature sets, such as the EO\( N \) even-odd quadrature [10], \( EO_N \) equal weight quadrature [11], and \( P_{N \cdot T_N} \) Legendre-Chebyshev quadrature [9], were developed as an alternative for discretizing the continuous angular variation. After the computational grid, quadrature scheme, and medium properties are set, Eq. (2) can be solved using a control-volume marching procedure. For brevity, further details on DOM solution procedure are not repeated here, but are readily available in previous publications [16, 21].

**PF NORMALIZATION METHODS**

It is widely recognized that scattered energy should be accurately conserved after directional discretization, i.e.,

\[
E = \frac{1}{4\pi} \sum_{i=1}^{M} \phi^{i'} w^i = 1, \quad i' = 1, 2, ..., M
\]  

(6)

This condition is automatically conserved for isotropic scattering (\( \Phi = 1 \)). However, it becomes increasingly violated as scattering anisotropy increases, resulting in substantial computational errors in practical problems where scattering is always anisotropic. Additionally, non-conservation of scattered energy can cause iterative divergence of the ERT solution procedure, rendering simulation useless.

In order to ensure scattered energy conservation after DOM discretization, PF normalization is commonly implemented. The most common approach to ensure accurate satisfaction of Eq. (6) is to normalize the scattering PF using a directional averaging approach, as follows [13]:

\[
\Phi^{i'} = \Phi^{i'} s \left( \frac{1}{4\pi} \sum_{i=1}^{M} \phi^{i'} w^i \right)^{-1}
\]  

(7)

Normalization of the scattering PF by the inverse of the scattered energy summation will automatically guarantee the accurate conservation of Eq. (6). This normalization technique is referred to as “scattered energy averaging” for the remainder of this study.

In addition, awareness that the PF asymmetry factor \( g \) should remain unaltered after directional discretization in order to retain prescribed medium properties [16] has recently grown, i.e.,

\[
\frac{1}{4\pi} \sum_{i=1}^{M} \phi^{i'} w^i \cos \Theta^{i'} = g, \quad i' = 1, 2, ..., M
\]  

(8)

where \( \Theta^{i'} \) is the scattering angle between discrete directions \( \hat{s}^i \) and \( \hat{s}^{i'} \). While scattered energy averaging of Eq. (7) is able to conserve Eq. (6), Eq. (8) remains non-conserved after substitution of the normalized PF. Non-conservation of Eq. (8) after directional discretization results in angular false scattering errors [21,22], which can critically impact radiation transfer results.

Ideally and mathematically, both Eqs. (6) and (8) should be simultaneously satisfied after directional discretization. In 2012, Hunter and Guo [16] published the first technique that can satisfy both constraints. In their 2012 approach, the PF values are normalized as follows:

\[
\Phi^{i'} = \left( 1 + A^{i'} \right) \Phi^{i'}
\]  

(9)

where the normalization parameter matrix \( A^{i'} \) is determined such that \( \Phi^{i'} \) satisfies Eqs. (6) and (8) simultaneously. DOM radiation transfer results generated using Eq. (9) have been shown to accurately conform to both FVM and Monte Carlo (MC) predictions [21].

Mishchenko et al. [19], and later Kamdem Tagne [20], introduced simple PF normalization approaches, through which either scattered energy or asymmetry factor conservation was achieved solely via normalization of the forward scattering term \( \Phi^{i'} \), as follows:

\[
\Phi^{i'} = \left( 1 + A^{i} \right) \Phi^{i'}
\]  

(10)

where \( A^{i} \) is the forward-scattering normalization vector parameter, expressed as follows for discrete direction \( \hat{s}^{i} \):

\[
A^{i} = \left[ 4\pi - \sum_{i=1}^{M} \phi^{i'} w^i \right] / \phi^{i'} w^i
\]

(11a)

\[
A^{i} = \left[ 4\pi g - \sum_{i=1}^{M} \phi^{i'} w^i \cos \Theta^{i'} \right] / \phi^{i'} w^i
\]

(11b)
Eq. (11a) lists the normalization parameters for Mishchenko et al.’s technique that will accurately conserve scattered energy, and Eq. (11b) lists the normalization parameters for Kamdem Tagne’s technique that will accurately conserve asymmetry factor. While these methods (referred to as Mishchenko E and Kamdem Tagne g hereafter) are simple in nature, neither one can concurrently conserve both $E$ and $g$.

More recently, Hunter and Guo [22] developed a simple normalization technique for HG PFs, drawing on the approach used by both Mishchenko E and Kamdem Tagne g. In order to conserve both scattered energy and asymmetry factor simultaneously, the backward-scattering term $\Phi_i^{\overline{r}^{-}}$ can be normalized in addition to the forward-scattering term $\Phi_i^{\overline{r}^{+}}$. Normalization of just these two terms allows for both of the critical constraints to be satisfied, while retaining the majority of PF values. Applying this concept, the normalized values of the forward and backward scattering PF terms can be expressed as follows:

$$\Phi_i^{\overline{r}^{+}} = (1 + A^i)\Phi_i^{\overline{r}^{+}}$$  \(12a\)

$$\Phi_i^{\overline{r}^{-}} = (1 + B^{\overline{r}^{-}})\Phi_i^{\overline{r}^{-}}$$  \(12b\)

where $A^i$ is the forward scattering normalization vector parameter, $B^{\overline{r}^{-}}$ is the backward scattering normalization vector parameter, and the superscript $\overline{r}$ refers to the direction directly opposite from $\overline{r}$. The values of the forward- and backward-scattering normalization parameters can be expressed as follows, for discrete direction $\overline{s}^i$:

$$A^i = \frac{1}{2\Phi_i^{\overline{r}^{+}}w_i^{\overline{r}^{+}}} \left[ 4\pi(1 + g) - \sum_{t=1}^{N} \Phi_i^{\overline{r}^{+}}w_i^{\overline{r}^{+}}(1 + \cos \Theta_i^{\overline{r}^{+}}) \right]$$  \(13a\)

$$B^{\overline{r}^{-}} = \frac{1}{2\Phi_i^{\overline{r}^{-}}w_i^{\overline{r}^{-}}} \left[ 4\pi(1 - g) + \sum_{t=1}^{N} \Phi_i^{\overline{r}^{-}}w_i^{\overline{r}^{-}}(\cos \Theta_i^{\overline{r}^{-}} - 1) \right]$$  \(13b\)

Use of this simple technique is able to accurately conserve both scattered energy and PF asymmetry factor simultaneously without requiring manipulation of a potentially cumbersome normalization matrix. Since this technique is published in 2014, it is referred to as Hunter and Guo’s 2014 technique hereafter.

### RESULTS AND DISCUSSION

As a means of comparing the five PF normalization techniques outlined in the previous section (scattered energy averaging, Mishchenko E, Kamdem Tagne g, Hunter and Guo 2012 and 2014), a detailed examination of the impact of PF normalization on radiation transfer computation is presented here for both HG and Legendre PFs. The workstation used in this study is a Dell Optiplex 780, with an Intel Core 2 processor and 4.0 GB of RAM. DOM radiation transfer results were generated in FORTRAN. For all normalization techniques except Hunter and Guo’s 2012, all calculations were directly performed in the FORTRAN environment. For results generated using Hunter and Guo’s 2012 technique, the normalization matrix $A_i^i$ in Eq. (9) was determined using MATLAB’s built-in least-squares approximation solver. Convergence times and required CPU memory for the five normalization techniques are quite similar.

The lack of either scattered energy or asymmetry factor conservation for the various normalization techniques is presented in Table 1 for three typical values of HG asymmetry factor: $g = 0.6000, 0.8000$, and $0.9300$. In said table, scattered energy conservation and asymmetry factor values are tabulated for DOM $P_N$-$T_N$ quadrature indices of $N = 4, 6, 8, 12$, and 16 corresponding to $M = 24, 48, 80, 168$, and 288 discrete directions, respectively. As a means of validating PF normalization necessity, non-normalized scattered energy and asymmetry factor conservation values are also presented. For both scattered energy averaging and Mishchenko E, only discretized $g$ values are presented, as scattered energy is conserved. Conversely, for Kamdem Tagne g, only scattered energy conservation is presented. Percent differences in both $E$ and in $(1-g)$ are also listed, in order to gauge the significance of lack of parameter conservation. Change in scattering effect is manifested in the difference in reduced scattering coefficient, or say, $(1-g)$ according to the isotropic scaling law [23], and not $g$ itself, and thus these values are presented in order to more properly gauge the impact of improper conservation. As both Hunter and Guo’s 2012 and 2014 approaches conserve both quantities simultaneously, their data are not tabulated.

For $g = 0.6000$, lack of PF normalization results in significant distortions in both $E$ and $g$ for small direction number ($N < 8$). As quadrature level is increased, discrepancies dramatically reduce, with no appreciable difference in either quantity witnessed for $N = 16$. As scattering becomes more highly anisotropic, drastic differences in both $E$ and $g$ are observed for all direction numbers. For extreme forward scattering ($g = 0.9300$), differences of $>80\%$ in both quantities are witnessed for $N = 16$, indicating the absolute necessity of PF normalization.
Application of scattered energy averaging is able to accurately conserve energy in the system for all cases. However, as angular false scattering errors in radiation transfer have been directly tied to asymmetry factor distortion after angular discretization [21], changes in \( g \) must be analyzed. Minimal discrepancies in \( g \) are observed for prescribed \( g = 0.6000 \), with differences in scattering effect of \( >0.5\% \) witnessed only for \( N = 4 \) and 6. An increase in the prescribed asymmetry factor to \( g = 0.8000 \) results in scattering effect changes of \( >10\% \) for \( N < 12 \). Further, for the extreme case \( (g = 0.9300) \), differences in scattering effect (1-\( g \)) exceed 45\% for all quadratures, with discretized asymmetry factors of \( g = 0.9965, 0.9925, 0.9873, 0.9746, \) and 0.9617 vastly overpredicting the prescribed value for all five quadrature orders, respectively.

Mishchenko’s \( E \) normalization is able to effectively conserve scattered energy while preserving the majority of the original PF values. The simple notion of normalizing only the forward-scattering terms leads to much smaller errors in discretized \( E \) than seen after implementation of scattered energy averaging for all quadratures. However, for the extreme case, \( >5\% \) changes are still observed for \( N < 12 \), with differences of 16.29\% and 10.43\% occurring for \( N = 4 \) and 6, respectively. Thus, while discrepancies in (1-\( g \)) are reduced using this simple normalization over those previously witnessed for scattered energy averaging, they may still be significant.

Kamdem Tagné’s \( g \) normalization is able to accurately conserve \( g \) after angular discretization, unlike the previously discussed techniques, meaning that angular false scattering errors should be minimized. However, scattered energy conservation is not guaranteed using this approach. Discrepancies in \( E \) using Kamdem Tagné’s \( g \) normalization are not as large as the errors in \( g \) for the two \( E \)-conserving approaches. For \( g = 0.6000 \), energy is only slightly non-conserved for low quadratures. Increase in prescribed asymmetry factor results in slight increases in energy non-conservation, but the values remain minimal (less than 1\% for all quadratures except \( N = 4 \)). Although these deviations are of low magnitude, accumulation of errors during computation due to lack of energy conservation can be appreciable and/or lead to ERT iterative divergence, and thus they should not be ignored.

Figure 1a plots discretized HG PF values versus scattering angle cosine generated with the five normalization techniques for the \( P_{8T_8} \) quadrature \( (M = 80) \) with \( g = 0.9300 \). As a comparison, the theoretical HG PF values are also plotted. For scattered energy averaging, the discretized PF values deviate greatly from the theoretical, corresponding to alteration of \( g \) from the prescribed value \( g = 0.9300 \) to 0.9873 after angular discretization. In essence, an entirely new PF is created, corresponding to the discretized \( g \) value. For Mishchenko’s \( E \) and Kamdem Tagné’s \( g \) technique, all PF values except those at \( \cos \Theta = 1 \) remain unaltered from the theoretical. Hunter and Guo’s 2014 technique is similar, with an additional deviation witnessed at \( \cos \Theta = -1 \), due to the addition of backward parameters. Hunter and Guo’s 2012 normalization slightly alters all PF values in the system, although the deviations appear minimal, retaining the accurate conformity of discretized PF values to the theoretical PF at most locations.

A common element that exists for all five normalization techniques is the normalization of the forward-scattering PF term. Normalization of this term is necessary to artificially reduce the forward-scattering peak such that \( E, g \), or both are conserved. Figure 1b presents the values of the forward-scattering normalization parameters for the \( P_{8T_8} \) quadrature.
and HG $g = 0.9300$ for the five normalization approaches. The parameters are only presented for directions in the principal octant, due to DOM directional symmetry. For Hunter and Guo’s 2014, Mishchenko’s E, and Kamdem Tagne’s $g$ normalizations, values of the forward-scattering normalization parameter $A_1^f$ are presented. For Hunter and Guo’s 2012 normalization, the parameter $A_1^f A_1$ is plotted. For scattered energy averaging, the value shown is the value of the inverse normalization, the $P$ parameter.

$$\text{normalizations, values of the forward-scattering normalization parameter } A_1^f \text{ are presented. For Hunter and Guo’s 2012 normalization, the parameter } A_1^f A_1 \text{ is plotted. For scattered energy averaging, the value shown is the value of the inverse normalization, the } P \text{ parameter.}$$

As seen in Fig. 1b, the normalization parameters generated using the three simpler approaches (Hunter and Guo’s 2014, Mishchenko’s E, and Kamdem Tagne’s $g$) are nearly identical, although each approach conserves a different combination of $E$ and $g$. The forward-scattering normalization parameters calculated using Hunter and Guo’s 2012 technique are slightly higher in absolute value than the three previously addressed approaches. This is due to the fact that, unlike the three techniques which simply normalize the forward peak, every PF term is normalized in Hunter and Guo’s 2012 approach. Thus, the drop in forward-scattering PF term is counteracted by slight increases in other PF values. While scattered energy averaging more accurately retains the forward-scattering peak value after normalization, the alteration of the remaining PF values witnessed in Fig. 1a is significant.

The data in Table 1 and Figures 1(a-b) provide a preliminary indication as to the importance of proper PF normalization to conserve $E$ and $g$. In order to further investigate the impact of the five normalization techniques on radiation transfer predictions, a benchmark test problem involving steady-state radiation transfer in a cubic enclosure of edge length $L$ is examined. The cubic enclosure houses a purely scattering ($\omega = 1.0$) medium, which scatters radiant energy anisotropically with $g = 0.9300$. To ensure invariance with spatial grid refinement, a staggered spatial grid of $(N_x \times N_y \times N_z) = 27 \times 27 \times 27$ is applied, and the positive differencing scheme is implemented. The spatial coordinates are non-dimensionalized in the following manner: $x' = x/L$, $y' = y/L$, and $z' = z/L$. The medium and enclosure walls are cold and black, except for the wall at $z' = 0$, which is taken as a diffuse emitter with unity emissive power.

Figures 2(a-b) present non-dimensional heat flux $Q$ at the centerline of the wall opposite the diffuse emitter, i.e. $Q(x', y' = 0.5, z' = 1.0)$, generated using the DOM with the five previously discussed normalization techniques. The optical thickness of the medium is taken to be $\tau = (\sigma_a + \sigma_s)L = 10.0$, corresponding to a reduced optical thickness $(1-g)\tau = 0.70$. Results are presented for the $P_9T_8$ quadrature with $M = 80$ discrete directions in Figure 2a, and with $M = 288$ discrete directions in Figure 2b. As a means of validation, heat fluxes generated using the FVM are also plotted. The FVM profiles were generated using the FT-FVM quadrature set [25] with $M = 2400$ discrete directions. At this directional order, scattered energy and asymmetry factor are conserved within 0.07% and 0.05%, respectively, thereby minimizing errors due to $E$ or $g$ non-conservation.

For $M = 80$, heat flux generated using scattered energy averaging normalization deviates greatly from the remaining four normalization techniques, due to drastic alteration of $g$ from 0.9300 to 0.9873. Overpredictions in heat flux as compared to high-order FVM range between 28-64%, indicating the importance of accurate $g$ conservatism. The remaining four normalizations produce heat fluxes that conform to high-order FVM within 8%. Use of Mishchenko’s E normalization alters the discretized asymmetry factor to $g = 0.9300$.
0.9348, resulting in a heat flux that overpredicts Hunter and Guo’s 2012 normalization (which conserves both quantities) by ~3%. Conversely, the lack of energy conservation in Kamdem Tagne’s g normalization results in an underprediction of ~4%. The average difference between Hunter and Guo’s 2012 and 2014 normalizations, which both conserve both quantities, is less than 0.5%.

Increase in discrete direction number to $M = 288$ in Figure 2b shows an even greater minimization of error between all normalization techniques. Scattered energy averaging error decreases from that witnessed in Figure 2a (due to a reduction in discretized $g$ to 0.9617), although overpredictions of between 20-32% still persist. Mishchenko’s E ($g = 0.9313$) and Kamdem Tagne’s g (E = 0.9987) normalizations differ slightly from Hunter and Guo’s 2012 and 2014 normalizations, with average differences of ~1%, while both of Hunter and Guo’s normalizations conform to one another within 0.08%. The four non-averaging approaches conform accurately to extremely high-order FVM, while providing a distinct advantage in computational efficiency. The FVM solution with $M = 2400$ required ~37 hours to converge, while the equally accurate $M = 288$ DOM results only required ~18 minutes (a reduction of over 99%).

An analysis of the impact of normalization technique on radiation transfer in an optically-thin medium ($\tau = 1.0, (1 - g) \tau = 0.07$) is illustrated in Figure 3. DOM heat fluxes are presented for $M = 48, 80$, and 288 discrete directions, respectively, while high-order FVM heat flux with $M = 2400$ is again plotted for comparison. For the lower-order DOM ($M = 48$ and 80), the appearance of substantial unrealistic bumps in heat flux profiles due to ray effect [7] is observed. Increasing direction number to $M = 288$ reduces ray effect by more accurately approximating the angular domain. For all directional orders, heat fluxes generated using scattered energy averaging again overpredict the remaining four normalizations due to large non-conservation of $g$. However, due to the reduction in total scattering events inherent with a reduction in optical thickness, angular false scattering errors due to non-conservation of $g$ are less substantial than those witnessed in Figures 2(a-b). The remaining four normalization techniques produce heat fluxes that conform to high-order FVM within an average of 1% for $M = 288$, indicating that minimal discrepancies in E or $g$ are negligible for optically thinner media. In general, for optically thinner media, ray effect appears to be the dominant source of error for coarse angular discretization.
For applications involving turbid media, such as laser light transport in biological tissues, it is common for the reduced optical thickness to be $>> 1$. To this end, Figure 4a presents an analysis of the impact of the five normalization techniques on radiation transfer in an optically-thick ($\tau = 100.0$) medium with HG $g = 0.9300$. The reduced optical thickness for this case is $(1 - g)\tau = 7.0$. Heat fluxes are plotted for both $M = 80$ and 288 discrete directions. For validation purposes, FVM heat flux generated using $M = 840$ discrete directions (discretized $g = 0.9280$) is presented. Increase in FVM direction number can improve discretized asymmetry factor, albeit with a substantial added computational cost. Use of $M = 2400$ discrete directions with $\tau = 100.0$ would have required roughly 3 weeks to converge, based on the number of required iterations and the elapsed time to complete one iteration. Reducing the number of directions to $M = 840$ resulted in a converged solution in ~32 hours.

Heat fluxes generated using scattered energy averaging vastly overpredict the FVM solution, with maximum overpredictions of ~650% and 160% corresponding to alteration of prescribed $g$ to 0.9873 and 0.9617 for $M = 80$ and 288, respectively. Increase in optical thickness greatly increases error due to lack of $g$ conservation, due to the large increase in medium scattering events. When Mishchenko’s $E$ normalization is applied, and $g$ is more accurately conserved (discretized $g = 0.9348$ and 0.9313 for the two $M$, respectively), errors in heat flux decrease dramatically. As seen in Figure 4b, which plots the percentage difference in DOM heat flux $Q(x^*, y^* = 0.5, z^* = 1.0)$ from high-order FVM, the error after application of Mishchenko’s $E$ normalization ranges between 7-13% for $M = 80$ and 3-7% for $M = 288$, indicating that small discrepancies in $g$ do have an impact, albeit rather small.

When Kamdem Tagné’s $g$ normalization is applied, much larger errors are witnessed than for Mishchenko’s $E$ normalization. The small discrepancies in scattered energy conservation ($E = 0.9952$ and 0.9987) result in large underpredictions in heat flux of ~60% and ~20% for the two direction numbers, respectively. This indicates the absolute importance of conserving scattered energy accurately in order to obtain accurate ERT solutions in practical participating media. When Hunter and Guo’s 2012 and 2014 techniques are implemented, both quantities are conserved, and errors with respect to high-order FVM range between 2-4% for $M = 80$ and 1-3% for $M = 288$. Additionally, both of Hunter and Guo’s techniques conform accurately to one another. Conformity to one another, as well as high-order FVM, gives confidence that both approaches are able to accurately solve the ERT numerically via proper conservation of both $E$ and $g$ simultaneously. Both approaches are highly applicable for HG phase-functions, although the mathematical simplicity inherent with Hunter and Guo’s 2014 technique may make it more desirable for use.

In addition to analyzing HG PF’s, it is necessary to investigate the impact of normalization on general Legendre PF’s, as these functions are more representative of the highly-oscillatory Mie scattering phase function. Figure 5a and plots discretized Legendre PF values versus scattering angle cosine generated with the five normalization techniques using the E0s quadrature for a representative Legendre PF. The Legendre PF chosen for this analysis is a 26-term PF with $g = C_1/3 = 0.8189$, whose Mie coefficients are put forth by Lee and Buckius [24]. The theoretical PF values are plotted as a comparison.

As seen in Figure 1a for the HG PF, the discretized values obtained after scattered energy averaging are highly skewed from the theoretical values, due to a distortion in $g$ to 0.9024.
Analysis of this critical issue is further examined in Figure 5b, in which the minimum values of forward-scattering parameter (or $A^f$ for Hunter and Guo’s 2012 technique) for the $g = 0.8189$ PF with $M = 80$ are plotted. For all 10 principal octant directions, forward parameters for Mishchenko E, Kamdem Tagne g, and Hunter and Guo’s 2014 techniques are nearly identical, with parameters of less than -1 occurring for three directions (negative PF values). For all but those three directions, Hunter and Guo’s 2012 technique produces nearly identical forward-parameters to the other three techniques. As seen in Figs. 1 and 2, both Hunter and Guo’s 2012 technique and scattered energy averaging are able to avoid the critical issue of negative PF values, due to compensation via normalization of all other PF values.

The appearance of negative PF values is potentially a disastrous issue for accurate radiation transfer computation. During simulation, negative PF values can lead to negative intensity values in some cases. In such cases, the ERT solution diverges at the next iteration, and a convergent ERT solution cannot be obtained. As a means of correcting this issue, the positive scheme [2,6] for negative intensity correction can be implemented, in which any negative intensities are set to zero as soon as they are encountered.

Figure 6a compares non-dimensional DOM heat flux distributions $Q(x', y', z' = 1.0)$ calculated using the five normalization techniques. The optical thickness of the medium is $\tau = (\sigma_a + \sigma_s) L = 10.0$. Results are generated using the $EO_9$ quadrature and $g = 0.8189$ Legendre PF. For comparison purposes, FVM heat flux profiles generated using the FT$_{\gamma}$ FVM quadrature with $M = 840$ directions are also presented. Use of $M = 840$ allows for scattered energy conservation within $10^{-4}\%$ and asymmetry factor conservation within 0.1% without normalization.

For the results generated in Fig. 6a, negative intensities were encountered for the three simpler normalization techniques that correct only one or two PF terms, corresponding to the negative PF values shown in Fig. 5a, and thus negative intensity correction was required to obtain convergent ERT solutions. Hunter and Guo’s 2012 technique is taken as a validation basis for DOM results, as this negative issue does not occur. Heat fluxes generated using Hunter and Guo’s 2014 technique with negative-intensity correction conform accurately to those generated using Hunter and Guo’s 2012 technique, with differences of less than 1.5% observed at both $y'$. Profiles generated using Mishchenko E technique overpredict Hunter and Guo’s 2012 technique by ~3% (discretized $g = 0.8233$), while profiles generated using Kamdem Tagne’s g technique underpredict by ~5% (discretized $E = 0.9957$). When compared with higher-order FVM, these four techniques produce heat fluxes that differ by a maximum of 8.8% at $y' = 0.1$ and 5.4% at $y' = 0.5$, which are acceptable differences for $M = 80$. Conversely, profiles generated using scattered energy averaging vastly overpredict both FVM and the remaining DOM heat flux profiles (discretized $g = 0.9024$).

Figure 5: (a) Discretized phase-function values and (b) forward-scattering normalization parameters for various normalization approaches using the $P_8 T_8$ quadrature with Legendre $g = 0.8189$.
Due to the large absolute values of the backward parameters in Hunter and Guo’s 2014 technique (as seen in Fig. 5a at \(\cos \theta = -1\)), it is of interest to show normalized heat fluxes at the hot wall, or \(Q(x^*, y^*, z^* = 0.0)\), in order to determine if significant changes in radiant energy propagating back into the hot wall occur. From Fig. 6a, it is observed that results generated using Hunter and Guo’s 2014 technique conforms to Hunter and Guo’s 2012 technique and to higher-order FVM within 0.2% for all \(x^*\) and \(y^*\) locations examined. Heat fluxes generated with the other two simple normalization techniques differ by less than 0.5% at maximum, while large overpredictions are again witnessed for scattered energy averaging due to large \(g\) distortions after angular discretization.

The results in Figs. 6(a-b) indicate that even though the three one/two-term(s) correction techniques can result in negative Legendre PF values, diffuse radiation transfer predictions are largely unaffected as long as a negative-intensity correction is put in place. However, the necessity of negative-intensity correction is not ideal, and thus Hunter and Guo’s 2012 technique is preferred for use with Legendre PF, as the negative issue is avoided. It should be mentioned that results were similar for other representative Legendre PFs and DOM quadrature sets, and thus they are not presented here, for brevity.

**CONCLUSION**

A comparison of the five PF normalization techniques for determining accurate DOM radiation transfer solutions are presented and analyzed. The following important guidelines are summarized:

1. Hunter and Guo’s 2012 technique is applicable for both HG and Legendre PFs. This technique conserves both scattered energy and asymmetry factor simultaneously, resulting in accurate radiative transfer predictions as compared with high-order FVM.

2. Hunter and Guo’s 2014 technique is highly applicable for diffuse radiation with HG PFs, due to its accuracy and simplicity of implementation. For Legendre PFs, however, this technique may result in negative PF values, requiring negative intensity correction to obtain accurate solutions.

3. Both Mishchenko et al.’s and Kamdem Tagne’s techniques are prone to negative forward PF values with Legendre PFs, although no such issue occurs for the HG PFs. However, lack of either scattered energy or asymmetry factor conservation inherent in these two techniques can cause significant errors in radiative transfer predictions, especially for optically-thick media.

4. While the popular technique of scattered energy averaging is a simple method of conserving scattered energy, the inability to simultaneously conserve asymmetry factor leads to large radiation transfer errors due to angular false scattering, which increase dramatically with increasing optical thickness.

5. The occurrence of negative Legendre PF values after normalization appears only in one or few correction directions. Via application of negative-intensity correction (such as the positive intensity scheme), converged radiation transfer results can still be obtained with minimal error resulting from such negative values.

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