NORMALIZATION FOR ULTRAFAST RADIATIVE TRANSFER ANALYSIS WITH COLLIMATED IRRADIATION

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ABSTRACT

Normalization of the scattering phase function is applied to the transient discrete ordinates method for ultrafast radiative transfer analysis in a turbid medium subject to a normal collimated incidence. Previously, the authors have developed a normalization technique which accurately conserves both scattered energy and phase function asymmetry factor after directional discretization for the Heneyy-Greenstein phase function approximation in steady-state diffuse radiative transfer analysis. When collimated irradiation is considered, additional normalization must be applied to ensure that the collimated phase function also satisfies both scattered energy and asymmetry factor conservation. The authors’ technique is applied to both the diffuse and collimated components of scattering using the general Legendre polynomial phase function approximation for accurate and efficient ultrafast radiative transfer analysis. The impact of phase function normalization on both predicted heat fluxes and overall energy deposition in a model tissue cylinder is investigated for various phase functions and optical properties. A comparison is shown between the discrete ordinates method and the finite volume method. It is discovered that a lack of conservation of asymmetry factor for the collimated component of scattering causes over-predictions in both energy deposition and heat flux for highly anisotropic media.

INTRODUCTION

Many heat transfer applications, including the interaction of laser light with biological media during biomedical laser therapeutic applications [1-6], involve the transport of radiative energy through turbid media. In said applications, the contributions of radiative heat transfer can dominate over both conduction and convection. In order to accurately determine the contributions of radiative transfer, accurate solutions of the Equation of Radiative Transfer (ERT) are required. The ERT is difficult to solve analytically, save for simplified cases, due to its integro-differential nature. As such, numerical methods are a common solution method for ERT so that radiation intensities, heat fluxes, and other radiative properties can be calculated. In many cases (where the undertaken problem can be modeled as a cylindrical enclosure under laser beam irradiation), the three-dimensional ERT can be simplified to two-dimensions using axisymmetry. There have been many proposed methods to solve the ERT, including the finite volume method [7], radiation element method [8], and direct integration [9]. One of the more commonly used methods, due to the relative simplicity of numerical implementation, is the Discrete Ordinates Method (DOM), introduced by Carlson and Lathrop in 1968 [10].

Since the advent of ultrafast lasers, ultrafast radiative heat transfer in turbid media has attracted increasing attention and interest from researchers [4-6, 11-15]. In ultrafast transfer, the ERT is time-dependent and radiation propagates with the speed of light in ultrashort time scale [8]. This differs from conventional transient radiative heat transfer, in which the ERT is stationary while the boundary condition is time-dependent. The introduction of a time-dependent term in the ERT transforms the equation into a hyperbolic format, complicating the modeling of the already-sophisticated integro-differential ERT. Mitra and Kumar examined short light pulse transport through one-dimensional scattering–absorbing media using different approximate mathematical models [4] and Guo and Kumar developed the transient DOM to solve the time-dependent ERT in two- and three-dimensional geometries [11,12]. Guo and co-authors further examined use of the transient DOM for applications such as laser-tissue welding and soldering and hyperthermia therapy [5,6], laser ablation of cancerous cells from surrounding healthy tissues [13], and optical tomography [14]. More recently, Akamatsu and Guo...
used Duhamel’s superposition theorem [12] to perform ultrafast radiative transfer analysis on a highly-scattering media subject to pulse train irradiation.

In turbid media, the scattering of light can be highly anisotropic. While the DOM can be used to solve the intensity field for turbid media, caution must be taken to ensure the conservation of scattered energy after directional discretization. When scattering is highly anisotropic, directional discretization using the DOM produces significant deviations in the conservation of scattered energy, leading to distorted and false results. To ensure that scattered energy is accurately conserved, phase function normalization is generally adopted. Previous techniques, including those by Kim and Lee [16] and Wiscombe [17], have been formulated to accurately conserve scattered energy after DOM discretization, regardless of DOM quadrature scheme. However, a more recent publication by Boulet et al. [18] concluded that even scattered energy was conserved accurately using these techniques, the overall phase function asymmetry factor and phase function shape were significantly altered for strong-forward scattering functions. A comparison to results generated using the Monte Carlo method further demonstrated that phase function shape distortion due to the specific normalization techniques skewed heat flux profiles significantly, establishing the need for accurate conservation of asymmetry factor.

To address this phenomenon, Hunter and Guo [19,20] developed a new phase function normalization technique, which was formulated to simultaneously conserve both phase function asymmetry factor and scattered energy. They confirmed the issue of phase function distortion, and applied their new normalization technique to diffusely-scattering media in order to mandate the necessity of conserving phase function asymmetry factor when determining medium heat fluxes. They applied this technique both to the DOM [19] and the Finite Volume Method (FVM) [20], noting that implementation of their normalization technique eliminates the need for splitting solid angles in the FVM into many sub-angles to specifically conserve scattered energy. The previous works by Hunter and Guo, however, only considered cases under diffuse incidence. When collimated incidence is concerned, the collimated component of the phase function must be treated independently from the diffuse component, leading to the necessity of determining the effect of phase function normalization on the collimated scattering phase function. Moreover, the previous studies solely focused on normalization of Henyey-Greenstein phase function. It is important to examine the impact of said technique when implementing general Legendre scattering phase functions.

In this work, the transient DOM with phase function normalization is implemented to determine ultrafast radiative transfer in a biological tissue sample subject to a normal collimated incidence at the top boundary. Conservation of both scattered energy and asymmetry factor is investigated for both the diffuse and collimated phase functions. Results generated for a basic cylindrical enclosure are compared to both previously published results and FVM calculations for validation. Both steady-state and transient heat flux profiles, generated with and without phase function normalization, are compared for various Legendre polynomial phase functions to determine the impact of phase function normalization. The impact of asymmetry factor, medium depth, and normalization technique on energy deposition in the medium is also investigated. Finally, a new formulation of divergence of radiative heat flux for ultrafast transfer analysis, presented by Rath and Mahapatra [21], is discussed and clarified.

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A^1$</td>
<td>Diffuse normalization parameter</td>
</tr>
<tr>
<td>$A_c^{c_t}$</td>
<td>Collimated normalization parameter</td>
</tr>
<tr>
<td>$c$</td>
<td>Medium speed of light</td>
</tr>
<tr>
<td>$g$</td>
<td>Asymmetry factor</td>
</tr>
<tr>
<td>$H$</td>
<td>Height of cylindrical enclosure</td>
</tr>
<tr>
<td>$I$</td>
<td>Radiative intensity ($W/m^2sr$)</td>
</tr>
<tr>
<td>$M$</td>
<td>Total number of directions</td>
</tr>
<tr>
<td>$N_x,N_z$</td>
<td>Number of radial and axial control volumes</td>
</tr>
<tr>
<td>$Q$</td>
<td>Radiative heat flux ($W/m^2$)</td>
</tr>
<tr>
<td>$r$</td>
<td>Position vector</td>
</tr>
<tr>
<td>$r$</td>
<td>Radial location</td>
</tr>
<tr>
<td>$R$</td>
<td>Radius of cylindrical enclosure</td>
</tr>
<tr>
<td>$\tilde{s}$</td>
<td>Unit direction vector</td>
</tr>
<tr>
<td>$w$</td>
<td>Discrete direction weight</td>
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<tr>
<td>$z$</td>
<td>Axial location</td>
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**Greek Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$\beta$</td>
<td>Extinction coefficient, $=\sigma_a + \sigma_s (mm^{-1})$</td>
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<tr>
<td>$\mu,\eta,\xi$</td>
<td>Direction cosines</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Scattering phase function</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Normalized scattering phase function</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Azimuthal location (rad)</td>
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<tr>
<td>$\rho$</td>
<td>Reflectivity</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>Absorption coefficient (mm$^{-1}$)</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>Scattering coefficient (mm$^{-1}$)</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Optical thickness</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>Scattering angle (rad)</td>
</tr>
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<td>$\theta$</td>
<td>Polar location (rad)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Scattering albedo, $=\sigma_s/\beta$</td>
</tr>
</tbody>
</table>

**Subscripts**

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Blackbody
Diffuse
Specular
Boundary wall

Superscripts

' Radiation incident direction
l,l' Radiation directions
l' From direction l' into direction l

EQUATION OF RADIATIVE TRANSFER

The time-dependent ERT for a diffuse, gray and absorbing-emitting-scattering media can be expressed, in general vector notation, as follows [22]:

$$\frac{1}{c} \frac{\partial I(r, s, t)}{\partial t} + \hat{s} \cdot \nabla I(r, s, t) = -\beta I(r, s, t) + \sigma_a I_b(r, t)$$

$$+ \frac{\sigma_s}{4\pi} \int I(r, s', t) \phi(s', s) d\Omega'$$

(1)

In the preceding equation, the terms on the left-hand side represent the temporal and spatial gradients of radiative intensity, while the three terms on the right-hand side represent (1) intensity attenuation due to medium absorption and out-scattering, (2) intensity augmentation due to medium absorption, and (3) intensity augmentation due to radiant energy in-scattering.

For this study, consider an axisymmetric cylindrical sample of biological tissue, which both absorbs and anisotropically scatters radiant energy. The tissue sample has radius \( R \) and depth \( H \), and is subject to a normal collimated incidence with a magnitude of \( I_c \) at the top axial surface (\( z = 0 \)). Using the transient DOM, Eq. (1), for the diffuse component of intensity, can be expressed as follows for a discrete direction \( s' \) [5,23]:

$$\frac{1}{c} \frac{\partial I^t}{\partial t} + \frac{\mu^t}{r} \frac{\partial}{\partial r} [r I^t] - \frac{1}{r} \frac{\partial}{\partial \phi} [\eta^t I^t] + \xi^t \frac{\partial I^t}{\partial z} = -\beta I^t + \beta S^t$$

(2)

where the three discrete direction cosines \( \mu^t, \eta^t, \) and \( \xi^t \) correspond to the \( r, \phi, \) and \( z \) directions, respectively.

The radiative source term \( S^t \), for a case involving collimated irradiation, can be expanded into three different components [24]:

$$S^t = (1 - \omega) I_b + \frac{\omega}{4\pi} \sum_{i'=1}^M w^{i'i} \phi^{i'i} I^{i'i}$$

$$+ \frac{\omega}{4\pi} I_c \phi^{i'c} \exp(-\tau_z)$$

(3)

The first two terms correspond directly to the absorption and in-scattering intensity augmentation terms given in Eq. (1). Discretization using the transient DOM leads to the use of the discrete directional weights \( w^{i'i} \), which vary for all directions and must satisfy given moment constraints. The discrete diffuse scattering phase function \( \phi^{i'i} \) represents scattering between discrete directions \( \hat{s}^{i'} \) and \( \hat{s}^i \). The third term represents the propagation and scattering of the collimated incidence, which follows the exponential decay of the Beer-Lambert law. The optical depth \( \tau_z = \beta z \) ranges from \( \tau_0 = 0 \) at the collimated incident surface to \( \tau_H = \beta H \) at the opposite axial surface. Thus, the strength of the collimated incidence will decay as light penetrates further into the medium. The collimated (ballistic) scattering phase function \( \phi^{i'i} \) represents scattering between the direction of collimated incidence \( \hat{s}^{i'} \) and discrete direction \( \hat{s}^i \).

For an axisymmetric cylinder of biological tissue, three separate boundary conditions must be addressed. Due to the mismatch of the refractive indices of tissue and air, a Fresnel reflection boundary condition is imposed at the collimated incident surface (\( z = 0 \)). This surface is taken as a specular reflector, with the intensity emanating in direction \( \hat{s} \) from the surface calculated using the following relation [5,11]:

$$I^t \cdot \hat{n} = (1 - \rho_{w,s}) I_{bw} + \rho_{w,s} I^{l^{-i}}$$

(4)

The specular reflectivity of the wall is evaluated using Fresnel’s law as follows:

$$\rho_{w,s} = \left\{ \begin{array}{ll}
\frac{\tan^2(\theta^l - \theta^t)}{\tan^2(\theta^l + \theta^t)} & \theta^t < \theta_{cr} \\
1 & \theta^t \geq \theta_{cr}
\end{array} \right.$$  

(5)

where \( \theta^t \) is the angle of incidence of the incoming radiation, \( \theta^l \) is the corresponding angle of refraction (which can be determined using Snell’s Law), and \( \theta_{cr} = \sin^{-1}(n_{air}/n_{tissue}) \) is the critical angle. The first term in Eq. (4) represents blackbody emission, and the second term represents specular reflection, where \( I^{l^{-i}} \) is the intensity in direction \( \hat{s}^{-i} \), which has direction cosines \( \mu^{-i} = \mu^l \) and \( \xi^{-i} = -\xi^l \).

The boundary conditions at the bottom axial wall (\( z = H \)) and the radial side wall (\( r = R \)) are taken as diffuse reflectors. At the bottom wall, the emanating intensity can be calculated as a sum of blackbody emission and the reflection of both the collimated incidence and all incoming intensities:

$$I^t_{bw} = (1 - \rho_{w,d}) I_{bw} + \frac{\rho_{w,d}}{\pi} I_c \exp(-\tau_H) + \sum_{i'<l, \xi'<0} w^{i'i} I^{i'i} \xi^{i'i}$$

(6)

where the diffuse reflectivity \( \rho_{w,d} \) is taken to have a value of 0.5, due to the high scattering nature of turbid tissue [5,11].
the radial side wall, the condition can be written in a similar manner with proper manipulation of the direction cosines. For the specific case where the collimated incidence is normal to the top surface, the exponential term is negated because no direct portion of the collimated incidence can be absorbed by the radial wall. Finally, at the radial centerline (r = 0), an axisymmetric boundary condition (perfectly specular condition) is imposed.

Once the governing equation is solved to determine the overall medium intensity field, the incident radiation at any location in the medium can be calculated as follows [11,12]:

\[ G = \sum_{l=1}^{M} w^l I^l + I^c \exp (-\tau_x) \]  

(7)

The heat fluxes in both the radial and axial directions are determined as follows:

\[ Q_r = \sum_{l=1}^{M} w^l I^l \mu^l, \quad Q_z = \sum_{l=1}^{M} w^l I^l \xi^l + I^c \exp (-\tau_x) \]  

(8)

SCATTERING PHASE FUNCTION

In general, the Mie phase function is highly oscillatory [22]. Although it is possible to implement the exact Mie phase function in calculations, researchers commonly implement approximations to the Mie phase function in order to avoid computation difficulty and facilitate numerical implementation. One of the more common ways to approximate the Mie phase function is to use a finite series of Legendre polynomials [22,25]. Using the Legendre polynomial approximation after DOM discretization, the diffuse scattering phase function can be represented as follows, for the axisymmetric cylindrical medium:

\[ \Phi^{l'l'} = \frac{1}{2} \left[ \sum_{i=0}^{N} C_i P_i (\cos \Theta_1^{l'1'}) + \sum_{i=0}^{N} C_i P_i (\cos \Theta_2^{l'1'}) \right] \]  

(9)

where the Legendre coefficients \( C_i \) are fitting coefficients determined from Mie theory, and the scattering angles \( \Theta_1^{l'1'} \) and \( \Theta_2^{l'1'} \) between directions \( l' \) and \( l \) are calculated as:

\[
\begin{align*}
\cos \Theta_1^{l'1'} &= \mu'^l \mu^l + \eta^l \eta'^l + \xi^l \xi'^l \\
\cos \Theta_2^{l'1'} &= \mu'^l \mu^l - \eta^l \eta'^l + \xi^l \xi'^l 
\end{align*}
\]  

(10)

The necessity for two scattering angles comes from the axisymmetric nature of the problem. The direction cosine \( \eta^l \) is calculated using the relation \( \eta^l = \pm \sqrt{1 - (\mu'^l)^2 - (\xi'^l)^2} \), which leads to two possibilities for the sign of the product \( \eta'^l \eta^l \).

The collimated scattering phase function \( \Phi^{l'C'}l' \) can also be expressed using Legendre polynomials. For the normal collimated problem considered, \( \mu^C = 0, \xi^C = -1, \eta^C = 0 \). Thus, using both Eqs. (9-10) along with these direction cosines, \( \Phi^{l'C}l' \) can be expressed as follows:

\[ \Phi^{l'C}l' = -\sum_{i=0}^{N} C_i P_i \xi'^l \]  

(11)

For brevity, further details on discretization procedure and solution method using the DOM are not presented here, but can be found in great detail in textbook [22] and journal papers [5, 11, 12].

PHASE FUNCTION NORMALIZATION

In real situations involving scattering of radiant energy, said energy is always exactly conserved due to the laws of thermodynamics. However, when the DOM is used to solve the ERT, scattered energy may not be accurately conserved after directional discretization. In order to ensure that scattered energy is accurately conserved after discretization, the diffuse scattering phase function must satisfy the following summation for each discrete direction \( s^l \) [22]:

\[ \frac{1}{4\pi} \sum_{i=1}^{M} \Phi^{l's^l} w^l = 1 \]  

(12)

In addition to the conservation of scattered energy, the overall asymmetry factor \( g \) of the scattering phase function should not be altered after directional discretization. In order to preserve \( g \), and thus preserve the overall shape of the phase function, the following summation must also be satisfied, for a discrete direction \( s^l' \):

\[ \frac{1}{4\pi} \sum_{i=1}^{M} \Phi^{l's^l'} \cos(\Theta^{l's^l'}) w^l = g \]  

(13)

Using a similar treatment as in Eqs. (9-10) for the separate scattering angles, the multiplication of \( \Phi^{l's^l'} \cos(\Theta^{l's^l'}) \) can be expanded as follows:

\[ \Phi^{l's^l'} \cos(\Theta^{l's^l'}) = \frac{1}{2} \left[ \sum_{i=0}^{N} C_i P_i (\cos \Theta_1^{l's^l'}) \cos \Theta_1^{l's^l'} + \sum_{i=0}^{N} C_i P_i (\cos \Theta_2^{l's^l'}) \cos \Theta_2^{l's^l'} \right] \]  

(14)

The constraints given by Eqs. (12-13) are not always satisfied after directional discretization, especially as scattering becomes highly anisotropic. In order to correct this problem, and assure conservation of scattered energy and asymmetry factor, phase function normalization is implemented. Previously published normalization techniques, specifically
those of Kim and Lee [16], Liu et al. [26] and Wiscombe [17] were created to specifically address scattered energy conservation, assuring that the relation of Eq. (12) is conserved after directional discretization. However, detailed investigations by both Boulet et al. [18] and Hunter and Guo [19,20] showed that these techniques do not conserve phase function asymmetry factor, leading to inaccurate results.

Hunter and Guo [19,20] recently formulated a new normalization procedure, designed to guarantee accurate conservation of scattered energy and asymmetry factor simultaneously after directional discretization. The diffuse scattering phase function can be normalized in the following manner:

$$\tilde{\Phi}^{\ell l} = \left(1 + A^{\ell l}\right) \Phi^{\ell l}$$  \hspace{1cm} (15)

where $A^{\ell l}$ are normalization parameters that are determined such that $\tilde{\Phi}^{\ell l}$ satisfies the conservation constraints given by Eqs. (12-13). In addition, due to the fact that scattering phase function depends solely on the angle between two directions, $\tilde{\Phi}^{\ell l}$ also satisfies a symmetry constraint, i.e. $\tilde{\Phi}^{\ell l} = \tilde{\Phi}^{\ell l'}$. The normalization parameters can be determined using QR decomposition [27], or through a pseudo-inverse technique, as described in previous works by the authors [19,20]. In the authors’ previous works, the normalization technique was applied to cases involving diffuse irradiation to witness the impact of normalization on steady-state heat fluxes using both the DOM and FVM for ERT discretization. The presented results mandated the necessity for conserving both scattered energy and asymmetry factor simultaneously for cases where scattering was highly anisotropic.

For cases involving collimated incidence, special care must be taken when considering the collimated scattering phase function $\Phi^{\ell c l}$. The collimated scattering phase function must also satisfy the conservation of scattered energy and asymmetry factor conditions, namely

$$\frac{1}{4\pi} \sum_{i=1}^{M} \Phi^{\ell c l} w^{l} = 1, \quad \frac{1}{4\pi} \sum_{i=1}^{M} \Phi^{\ell c l} w^{l}(-\xi^{l}) = g$$  \hspace{1cm} (16)

It is important to note that the normalization of the collimated scattering phase function is independent of the normalization of the diffuse scattering phase function, leading to the possibility that one can be normalized without the other. This is due to the fact that the fixed direction of collimated incidence $\xi^{c}$ may be independent of the discrete directions in the chosen DOM quadrature scheme. Due to this fact, the normalization procedure for the diffuse scattering phase function will not impact the collimated phase function, unless the direction of collimated incidence exactly lines up with one of the discrete directions inherent in the DOM scheme.

Applying the technique of Hunter and Guo, the normalized collimated scattering phase function can be calculated as follows:

$$\tilde{\Phi}^{\ell c l} = \left(1 + A^{\ell c l}\right) \Phi^{\ell c l}$$  \hspace{1cm} (17)

where $\tilde{\Phi}^{\ell c l}$ satisfies the two relations given in Eq. (16) as well as directional symmetry. Using Eq. (17) in conjunction with diffuse normalization will guarantee accurate overall conservation of scattered energy and phase function asymmetry factor.

In order to verify that scattered energy and asymmetry factor are accurately conserved after phase function normalization, conservation ratios for each quantity are listed in Table 1 for three different Legendre polynomial phase functions. The asymmetry factors for the three phase functions, which can be seen plotted against scattering angle in Figure 1, are $g = 0.669723, 0.84534$ and $0.927323$. The Mie coefficients for $g = 0.669723$ and $0.84534$ were presented by Kim and Lee [28], whereas the coefficients for $g = 0.927323$ were presented by Lee and Buckius [29]. The DOM $S_{16}$ quadrature (288 total discrete directions) was used, as it is the highest order quadrature available in the literature. Three separate cases are considered: (1) neither the diffuse or collimated phase functions are normalized, (2) only the diffuse phase function is normalized, and (3) both phase functions are normalized. The conservation ratios are determined by evaluating the summations in Eqs. (12-13) and (16) for each discrete direction. These summations are then averaged over all directions and compared to theoretical values.

When normalization is neglected altogether, deviations from unity are seen for both the diffuse and collimated phase functions when scattered energy conservation is considered. Deviations from unity increase with increasing asymmetry factor, conforming to results presented in Hunter and Guo’s previous publications [19,20]. The largest deviations are seen for the highly anisotropic case ($g = 0.927323$), with scattered energy conservation deviating up to 15% for the collimated phase function. While the diffuse phase function scattered energy conservation ratio is only 1.00721 for $g = 0.927323$, specific directional values calculated by Eq. (12) range from 0.8956 to 1.096. This distinct lack of conservation of energy for individual directions will greatly impact heat flux results.

<table>
<thead>
<tr>
<th>Prescribed $g$</th>
<th>No Norm.</th>
<th>Diffuse Norm. Only</th>
<th>Diffuse/Coll Norm.</th>
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<tr>
<td></td>
<td>Diffuse</td>
<td>Coll.</td>
<td>Diffuse</td>
</tr>
<tr>
<td>Conservation of Scattered Energy</td>
<td>0.669723</td>
<td>0.99942</td>
<td>1.00157</td>
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<tr>
<td>0.84534</td>
<td>0.99660</td>
<td>1.04184</td>
<td>1.00000</td>
</tr>
<tr>
<td>0.927323</td>
<td>1.00721</td>
<td>1.19818</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

| Conservation of Asymmetry Factor | 0.669723 | 0.99937 | 1.00424 | 1.00000 | 1.00424 | 1.00000 |
| 0.84534 | 0.99621 | 1.05141 | 1.00000 | 1.05141 | 1.00000 | 1.00000 |
| 0.927323 | 1.00893 | 1.15639 | 1.00000 | 1.15639 | 1.00000 | 1.00000 |
Similar results are seen when asymmetry factor conservation is considered. For $g = 0.927323$, the discretized value of the diffuse phase function asymmetry factor is altered to 0.9356 (0.89% increase), while the discretized value of the collimated phase function asymmetry factor is altered to 1.072 (15.63% increase). Hunter and Guo [19] previously showed that even slight changes in scattered energy and asymmetry factor conservation, for the diffuse case, can produce significant errors in intensity field and heat flux profiles due to the change in scattering effect inherent with minor changes in $g$.

When the diffuse scattering phase function is normalized, both scattered energy and phase function asymmetry factor are accurately conserved for all three Legendre polynomials. However, the conservation ratios for the collimated phase function $\Phi c$ are not impacted, due to the previously discussed normalization independence. Differences of up to 14.9% and 15.6% are still witnessed in the scattered energy and asymmetry factor conservation ratios, respectively. When the normalization procedure of Hunter and Guo is implemented for both the diffuse and collimated phase functions, accurate conservation of scattered energy and phase function asymmetry factor is witnessed for all phase functions.

![Figure 1: Legendre phase function distributions](image)

**RESULTS AND DISCUSSION**

All of the calculations and results for this study were performed using a Dell Optiplex 780 workstation, complete with an Intel Core 2 processor and 4.0 gigabytes of RAM. Other than validation with the FVM, the ERT was solved using the transient DOM (TDOM) with the $S_{16}$ quadrature for all cases. The DOM scheme was implemented using the FORTRAN computing language, and the normalization parameters were calculated in MATLAB due to the existence of built-in pseudo-inverse functions. For a given phase function, the parameters need only be determined once, as they have no dependence on medium properties. Once determined, the parameters are imported into FORTRAN and used in the DOM scheme.

Before investigating the impact of normalization on radiation propagation in a biological tissue sample, it is necessary to validate our code. To this end, the DOM algorithm was applied to a simple test case, as presented in Jendoubi et al. [24]. In said case, a cylindrical enclosure with aspect ratio ($R/H = 0.5$), subjected to a normal collimated incidence $I^c$ on the top surface ($z = 0$), was investigated. The medium was taken to be gray. The optical depth $\tau = \beta R$ and the scattering albedo of the medium were set as unity. An axisymmetric boundary condition was imposed at the radial centerline, and the radial and bottom walls of the enclosure are cold and diffuse blackbody emitters. For this analysis, the spatial grid considered was $(N_r \times N_z) = (40 \times 80)$, in order to have equal grid spacing in the radial and axial directions. Steady-state results were determined by neglecting the transient derivative in the ERT and then solving the governing equation using an iterative procedure. For all following results, the radial location $r$ and axial location $z$ are non-dimensionalized in the following way: $r' = r/R$, $z' = z/H$.

Figure 2 compares non-dimensional radial wall heat flux $Q_r(r' = 1)$ versus axial location $z'$ for three different Legendre phase functions ($g = 0.0$ (isotropic), 0.4 and 0.669723). The heat fluxes are normalized by the collimated incidence $I^c$. The profiles were generated using both the DOM and the FVM, and compared to results determined by Jendoubi [24]. It is important to note that phase function normalization was not applied for this validation, due to the fact that Hunter and Guo previously showed that normalization was not necessary in the DOM for asymmetry factors less than $g = 0.7$ when the $S_{16}$ quadrature was implemented [19]. For all three tested asymmetry factors, excellent agreement is seen between the calculated and published DOM results. Furthermore, results determined using the FVM conform accurately to the DOM results (maximum difference of 1.75% occurring for $g = 0.669723$). This validates that our code treats the collimated component properly, and comparison with the FVM provides added confidence in the results of the DOM algorithm.

Attention is now turned to the biological tissue sample subjected to normal, collimated incidence at the top surface (as described earlier). For this study, the radial and axial dimensions of the tissue sample were taken as $R = 5$ mm and $H = 2$ mm. The optical depth and scattering albedo were assumed to correspond to those of human dermis [30]: $\tau = \beta H = (18.97 \text{ mm}^{-1}) \times (2 \text{ mm}) = 37.94$, $\omega = 0.9858$. For all simulations forthwith, the spatial grid was taken as $(N_r \times N_z) = (150 \times 60)$, keeping the radial and axial step size equal. For all transient results, a non-dimensional time step of $\Delta t^* = c \Delta t / H = 0.00893$ was chosen, in order to make sure that the distance that light propagates between two successive time steps is less than the size of a single control volume [11]. The refractive index of tissue is taken as $1.4$, meaning that the speed of light $c$ in the medium can be calculated as 0.214 m/ns.
Fig. 2: Validation of DOM code with published DOM S14 results [24] and FVM: steady-state radial wall heat flux vs. axial location for a purely-scattering, optically thin medium.

Figure 3 depicts steady-state, non-dimensional radial wall heat flux vs. axial location for the three considered Legendre phase functions \((g = 0.669723, 0.84534 \text{ and } 0.927323)\). Profiles are generated for the three different normalization cases that were previously explained in the discussion surrounding Table 1. As asymmetry factor increases, radiant energy is scattered more strongly away from the top wall (the collimated source location), leading to decreases in heat flux near the top wall of the cylinder and increases near the bottom wall. As asymmetry factor increases, the axial location of the maximum heat flux increases due to the increase in forward-scattering strength. For example, the maximum magnitude of heat flux occurs at \(z^* = 0.15\) for \(g = 0.927323\) and at \(z^* = 0.075\) for \(g = 0.669723\). As one moves towards the bottom wall, the large optical thickness of tissue hinders radiant energy propagation, leading to smoothly decreasing heat flux profiles.

The impact of phase function normalization can be seen by investigating the three different normalization cases. For \(g = 0.669723\), profiles generated using no normalization, only diffuse phase function normalization, and full normalization produce nearly identical heat flux profiles (maximum difference of 0.5% seen between the non-normalized case and the fully normalized case). This again conforms to results presented by Hunter and Guo [19], who found that scattered energy and asymmetry factor are accurately conserved without normalization for this asymmetry factor. As asymmetry factor increases to \(g = 0.84534\), deviations start to appear. Using the fully normalized case as a basis, a maximum difference of 29.29% is witnessed when only the diffuse scattering phase function is normalized. However, when no normalization is considered, a maximum difference of 1770.94% is witnessed (so large that the curve does not appear on the plot). While normalization of the diffuse scattering phase function does improve the results, the lack of conservation of scattered energy and asymmetry factor of the collimated phase function does have a large impact on heat flux profiles.

Again using the fully normalized case as a basis, a maximum difference of 15.56% is seen when only the diffuse scattering phase function is normalized. However, when no normalization is considered, a maximum difference of 1770.94% is witnessed (so large that the curve does not appear on the plot). While normalization of the diffuse scattering phase function does improve the results, the lack of conservation of scattered energy and asymmetry factor of the collimated phase function does have a large impact on heat flux profiles.

Fig. 3: Steady-state radial wall heat flux vs. axial location for various Legendre phase functions and various normalization cases.

As a further validation, Figure 4 again depicts steady-state, non-dimensional radial wall heat flux vs. axial location. However, the heat flux profiles are calculated using the FVM and compared to the fully-normalized results for the DOM in Figure 3. Similar to the DOM, all three cases produce nearly identical profiles for \(g = 0.669723\). As \(g\) is increased, deviations again start to appear, but they are much less noticeable than in the DOM. For example, the maximum difference between the non-normalized and fully-normalized case for \(g = 0.927323\) is only 4.49%, compared to 1770.94% for the DOM. This is due to the fact that scattered energy and asymmetry factor are better conserved for highly anisotropic scattering using the FVM, a fact discussed by Boulet et al. [18] and Hunter and Guo [20]. There are still slight deviations in scattered energy and asymmetry factor conservation present when the FVM is not normalized [20], but they do not impact the results as greatly. When the fully-normalized FVM profiles are compared to the fully-normalized DOM profiles, an excellent agreement is witnessed. Maximum differences of 0.21%, 0.58%, and 1.61% are witnessed for \(g = 0.669723, 0.84534\), and 0.927323, respectively, giving confidence that the full normalization procedure is accurately treating the DOM results.

Fig. 4: Validation of FVM code with published DOM S14 results [24] and DOM: steady-state radial wall heat flux vs. axial location for a purely-scattering, optically thin medium.
Figure 5 depicts transient profiles of radial wall heat flux vs. axial location for $g = 0.927323$ at various non-dimensional times. Since the case where normalization was entirely neglected produced extremely distorted results, that case is neglected in all further results. Only the diffuse phase function normalization case and the fully normalized case are considered. As time increases, increases in both the overall magnitude of heat flux and the propagation depth of radiant energy into the medium are witnessed. At a small time ($\Delta t^* = 0.3125$), the heat flux near the bottom wall is negligible due to the large optical thickness of the medium. As the steady-state solution is reached at large time, energy has fully propagated through the entire axial range, leading to appreciable heat flux throughout the medium.

For all times considered, heat flux profiles generated with only the diffuse phase function normalized overpredict those using full normalization. The percentage difference between the two cases increases as one moves from the top wall to the bottom wall. At small time ($\Delta t^* = 0.3125$), the percentage difference ranges from 11.3% at the top wall to 16.7% at the bottom wall. At large time ($\Delta t^* = 6.25$), the percentage difference ranges from 12.9% at the top wall to 15.5% at the bottom wall, indicating that a lack of conservation of scattered energy and asymmetry factor for the collimated phase function skews the overall results by upwards of 10% at all times, reinforcing the idea that the collimated phase function must be normalized to generate accurate solutions.

Figure 6 depicts non-dimensional heat flux at the bottom wall $-Q_x(z^* = 1)$ versus radial location for various non-dimensional times for $g = 0.927323$. The negative sign in the heat flux is used to normalize the profiles from 0 to 1, and also indicates that the net direction of heat flux is into the wall. Due to the side wall being cold and diffusely reflecting (with $\rho_d = 0.5$), the heat fluxes attain a maximum value near the radial centerline and a minimum value at the radial wall. At small times, the magnitude of heat flux at the bottom wall is miniscule due to the lack of propagation of collimated incidence through the medium. At larger times, the impact of the reflecting radial wall becomes larger due to an increase of radiant energy scattering.
generated with full normalization. As time increases, the percentage difference between the profiles decreases from 16.7% at \( t^* = 0.3125 \) to 15.5% at \( t^* = 6.25 \), matching the results from Figure 5. The percentage difference is found to be invariant with radial location, which is consistent with the fact that the collimated source has no radial variation in this case.

In addition to analyzing the impact of collimated phase function normalization on radial and axial heat flux in biological tissue, a comparison of the absorbed radiant energy at different medium locations is performed. For biomedical applications, such as cancerous cell laser ablation, it is important to control the amount of radiant energy that is absorbed by surrounding healthy tissues to mitigate damage. The amount of radiant energy absorbed in the participating media, or energy deposition, can be calculated as follows:

\[
g_{rad} = \beta(1 - \omega)(G - 4\pi I_b)
\]

(18)

where \( g_{rad} \) is the volumetric radiative absorbed energy rate. For this analysis, the medium is cold, and \( g_{rad} \) can be nondimensionalized by the collimated intensity \( I^c \) and medium depth \( H \) in the following manner:

\[
g_{rad}^* = \frac{\beta(1 - \omega)(G)}{4\pi I^c / H}
\]

(19)

A comparison of the transient changes in volumetric radiative absorbed energy rate along the radial centerline is shown in Figure 7. The energy deposition is calculated for three axial locations \( (z^* = 0, 0.5, 1) \) along the radial centerline for \( g = 0.669723, 0.84534 \) and 0.927323. For all axial locations and asymmetry factors, the absorbed energy rate at the centerline increases as time increases. The absorbed energy rate near the top wall is much larger than at the midplane and bottom wall for small times, due to the proximity of the collimated incidence. At the top wall, increases in phase function asymmetry factor correspond to a decrease in the magnitude of absorbed energy due to the stronger scattering of light away from the surface. Conversely, increases in asymmetry factor correspond to increases in absorbed energy rate at the axial midplane and bottom wall.

As seen in previous results, the profiles generated with only diffuse normalization overpredict the profiles generated with full normalization for all asymmetry factors and axial locations. For both the axial midplane and bottom wall, percentage difference between diffuse normalized and fully normalized profiles decreases with increasing time. For \( g = 0.927323 \), the maximum percent difference decreases from 16.65% at \( t^* = 0.3125 \) to 15.43% at \( t^* = 6.25 \) for the bottom wall, and from 16.43% to 15.41% at the axial midplane for the same times. However, at the top wall, the opposite trend is seen, with the percent difference increasing from 6.28% to 8.47% for the same time range. In addition, the differences near the collimated source are always smaller than at other axial locations. Similar trends are seen for \( g = 0.84534 \), although the discrepancies are diminished due to the conservation ratio difference shown in Table 1.

Figure 7: Transient radial centerline absorbed energy rate versus nondimensional time for various axial locations and asymmetry factors for diffuse phase function normalization and full normalization

\[
\frac{1}{c} \frac{\delta}{\delta t} \int l(r, \hat{s}, t) d\Omega + \nabla \cdot \left[ \int l(r, \hat{s}, t) \hat{s} d\Omega \right] = -\beta \int l(r, \hat{s}, t) d\Omega + \sigma_a \int l_b(r, t) d\Omega + \sigma_s \int \Phi(\bar{s}', \hat{s}) d\Omega \quad (20a)
\]

Although the discrepancies are diminished due to the conservation ratio difference shown in Table 1.
Noting that:

\[
\int_{4\pi} l(r, \hat{s}, t) d\Omega = q_{rad} , \int_{4\pi} l_b(r, t) d\Omega = 4\pi I_b(r, t) , \int_{4\pi} \Phi(\hat{s'}, \hat{s}) d\Omega = 4\pi
\]  

Eq. (20a) can be simplified to the following form:

\[
\frac{1}{c} \frac{\delta}{\delta t} \int_{4\pi} l(r, \hat{s}, t) d\Omega + \nabla \cdot q_{rad} = -\beta \int_{4\pi} l(r, \hat{s}, t) d\Omega + 4\pi \beta (1 - \omega) I_b(r, t) + \beta \omega \int_{4\pi} l(r, \hat{s}', t) d\Omega' 
\]  

Using the definition for incident radiation:

\[
G(r, t) = \int_{4\pi} l(r, \hat{s}, t) d\Omega
\]

the final expression for the divergence of radiative heat flux for ultrafast radiative transfer, presented by Rath and Mahapatra [21], becomes

\[
\nabla \cdot q_{rad} = \beta (1 - \omega) (4\pi I_b - G) - \frac{1}{c} \frac{\partial G}{\partial t}
\]

If the temporal term is neglected, this equation reduces to the traditional formulation of divergence of radiative heat flux (the same as Eq. (18), or what we have denoted as \( g_{rad} \) previously, only with the opposite sign).

Rath and Mahapatra argued that in ultrafast radiative transfer, the time-derivative of incident radiation could not be neglected, and that the transient term in Eq. (22) directly contributes to the energy deposition in the medium, impacting the calculation of medium temperature using the energy equation. In the opinion of the authors, however, this term is not directly related to energy deposition. The authors believe that this term represents the propagation of radiant energy through the medium, which accounts for the amount of propagating energy that is "trapped" in a specific control volume at a specific time instant, which will travel to adjacent control volumes at subsequent time instants without being physically absorbed by the medium at that specific instance. This notion is further supported by the fact that while energy deposition should be identically zero for a purely scattering situation (\( \omega = 1 \)), the addition of the time-dependent term yields a non-zero value of divergence of radiative heat flux for time instants prior to steady-state.

To illustrate this phenomenon, Figure 9 plots the divergence of radiative heat flux at the centerline for various non-dimensional times with \( g = 0.927323 \) using both Rath and Mahapatra’s formulation of divergence of radiative heat flux and the traditional expression (\( g_{rad} \)). Profiles are generated using full normalization in order to ensure accuracy. When the traditional formulation is implemented (i.e., the propagation term is neglected), a general increase in energy deposition is seen with increasing time. At small time, energy deposition is higher at the top wall due to the presence of the collimated source, but negligible at greater medium depths. With an increase in time comes an increase in energy deposition at all axial locations (as radiant energy scatters through the medium) until a steady-state condition is reached. The results conform to our expectations, as the overall amount of energy absorbed by a cold medium should strictly increase with time.

When the propagation term is included in the formulation, however, a much different phenomenon is witnessed. Near to the collimated source, there is a significant spike in energy deposition due to the high magnitude of the propagation term at small times. As the problem approaches the steady-state condition, the contribution of the propagation term rapidly diminishes, leading to a decrease in energy deposition. In fact, at small times, a wave-like propagation of radiant energy is witnessed, with the large spike at small times appearing due to the high magnitude of radiant energy propagating away from the source. As time increases, radiant energy is further scattered in all directions, leading to a decrease in the overall magnitude of the spike produced by the transient term.
In this investigation, phase function normalization was applied to the transient DOM for ultrafast radiative transfer analysis in turbid media subject to collimated irradiation. The phase function normalization technique introduced by Hunter and Guo for the diffuse scattering phase function was extended and applied to the collimated scattering phase function to gauge the importance of normalization when a collimated source (such as incident laser light) is applied to a turbid medium. Conservation of scattered energy and asymmetry factor for both the diffuse and collimated phase functions were examined for three cases: (1) neither of the phase functions are normalized, (2) only the diffuse phase function is normalized, and (3) both the diffuse and collimated phase functions are normalized. It was shown that normalization of the diffuse phase function has no impact on the collimated phase function, and that further normalization was required to conserve scattered energy and asymmetry factor.

The DOM code used in this study was validated using both FVM results and previously published DOM results. Axial heat flux profiles determined for various asymmetry factors were compared for the three normalization cases. It was found that for $g = 0.669723$, no appreciable improvement was noticeable when normalization was considered. However, as asymmetry factor increased, lack of normalization of both the diffuse and collimated phase functions produced large discrepancies in heat flux profiles. Differences of up to 1770% were witnessed for $g = 0.927323$ when normalization was neglected altogether, and 15.56% when only the diffuse phase function was normalized. Normalization has a much lower impact on profiles generated with the FVM, but fully normalized FVM and DOM profiles match accurately with one another. When transient results were considered for $g = 0.927323$, the differences between full normalization and diffuse normalization only were in excess of 10% for all radial and axial locations and times.

The volumetric radiative absorbed energy rate (i.e., the local energy deposition) is also discussed. Transient energy deposition along the radial centerline was examined at three different medium axial locations, and it was found that a lack of collimated phase function normalization introduces discrepancies in energy deposition for highly anisotropic scattering. Axial location had a strong impact on the difference between absorbed energy calculations using only diffuse normalization and full normalization. The percentage difference between these profiles decreased with increasing time near the collimated source and increased with increasing time at the other axial locations for all times. All of the results presented dictate that normalization of both the diffuse and collimated phase functions is necessary to preserve scattered energy and asymmetry factor conservation after directional discretization when collimated incidence is introduced.

Finally, a new formulation of divergence of radiative heat flux for ultrafast radiative transfer, proposed by Rath and Mahapatra, is discussed. While the authors’ agree that the addition of a transient term to the divergence of radiative heat flux is applausive, they believe that this term is unrelated to local energy deposition. Rather, this term is a description of wave-like radiant energy propagation through the medium. When wave propagation is negligible, the transient term is eliminated and the divergence of radiative heat flux indeed represents local energy deposition.
REFERENCES


